THE

QUADRATURE OF THE CIRCLE

CORRESPONDENCE

BETWEEN

AN EMINENT MATHEMATICIAN

AND

JAMES SMITH, Esq.

(MEMBER OF THE MERSEY DOCKS AND HARBOUR BOARD,)

AUTHOR OF "THE QUESTION: ARE THERE ANY COMMENSURABLE RELATIONS BETWEEN A CIRCLE AND OTHER GEOMETRICAL FIGURES? ANSWERED BY A MEMBER OF THE BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE."

"*Strike, but Hear!*"

LONDON
SIMPKIN, MARSHALL & CO., STATIONERS' HALL COURT
EDINBURGH: OLIVER & BOYD.
MDCCCLXI.

EDWARD HOWELL, PRINTER, CHURCH STREET, LIVERPOOL.

CONTENTS.

LIST OF PLATES.

THE

QUADRATURE OF THE CIRCLE.

INTRODUCTION.

"No amount of attestation of innumerable and honest witnesses, would ever convince any one versed in mathematical and mechanical science, that a person had squared the circle, or discovered perpetual motion."—*Baden Powell, in Essays and Reviews, 8th Ed., page* 141.

"THE Quadrature of the Circle" is a problem which it has long been pronounced an impossibility to solve, on the authority of names, of such high standing in the scientific world both in ancient and modern times, that latterly, every man who has entertained the idea of attempting its solution has been regarded as a wild enthusiast. "*You may as well attempt to square the circle,*" has passed into a proverb, which is as familiar to the peasant as to the philosopher, and the former, would probably with just as little hesitation as the latter, arrive

at the conclusion, that the man who could imagine he had discovered its solution, was in a state of mind which hardly fitted him to be entrusted with the care of his own person. And yet, the solution of the problem is extremely simple after all. It would almost appear as if its very simplicity had been the grand obstacle which had hitherto stood in the way of its discovery.

"The British Association for the Advancement of Science" may assume infallibility, and authoritatively proclaim that the solution of the problem is impossible; and may consequently decline to permit the consideration of the subject to be introduced into their deliberations. Other learned Associations may add the weight of their authority by endorsing such a course of procedure. The Astronomer Royal of England *may make a display of his contempt* for any one who should venture to address him as an "authority" on the question. And yet, they must all ultimately succumb to the force of truth, however humiliating it may be to their pride to submit to the infliction.

The following correspondence arose out of a pamphlet (see Appendix A.) which I published about the period of the last meeting of the British Association at Oxford. At that meeting I distributed about 500 copies of the pamphlet among the *Savans* there assembled, and at the same time, I forwarded a copy to each member of the two Houses of Parliament. In this way I distributed

upwards of 1500 copies, among the most learned and best educated men in this country; and I did so, in the hope that some one of them would take up the consideration of this important question from my point of view, and afford me the opportunity of a candid and careful discussion of it. In this hope I have not been altogether disappointed.

My correspondent claims, and justly claims, to be an "authority" on this great question, having obtained the highest mathematical honors at the University of Oxford. He undertook to point out that (what he has been pleased to term) my highly ingenious reasoning in the pamphlet, rests on a fallacy. Our readers will judge, after perusing the correspondence, how far he has succeeded in accomplishing the task.

My theory is, that in every circle, *the circumference is exactly equal to three and one-eighth times its diameter; and the area exactly equal to three and one-eighth times the area of a square described on its radius;* and I have demonstrated these facts by a variety of diagrams, and could have adduced many more, had I thought it necessary.

I submit, that my correspondent has entirely failed to subvert this theory, and for the best of all reasons: "*It is one of the great truths of nature, which can admit of no doubt, and which it is not in the power of any man living to subvert.*" The latter sentence I quote from the letter of another correspondent, only recently received, who, in all his former communications had, with great

fairness and candour, conscientiously opposed me.

Previously to the twenty-ninth meeting of the British Association, at Aberdeen, in 1859, my enquiries had been confined to an examination of the relations existing between circles and squares, but even this limited condition of the enquiry had thoroughly satisfied me, as to the true relation between the diameter and circumference of a circle; and I resolved to bring the subject under the notice of the Association, of which I have been a member almost from the earliest period of its existence.

A short time before the meeting of the Association, I addressed a letter to the Honorary Secretary, informing him that it was my intention to attend the meeting, and that I purposed reading a Paper "*On the true Circumference and Area of a Circle.*" I received a reply from him, to say my Paper was placed on the books, and requesting me to inform him, on what day I should be prepared to read it. To this I replied, that I should be prepared to read it on the first day of the meeting of the Sections, if necessary, or on any subsequent day if more convenient to the Association.

I was told by several of my acquaintance that the Association would never give me a hearing, and that if I wished to spare myself considerable annoyance I had better stay at home. I felt that the subject was of too much importance to the interests of science, to justify me

in permitting it to be stifled, without any effort on my part to prevent it, and I resolved, at all risks, to attend the meeting of the Association.

I had the pleasure to hear His Royal Highness the Prince Consort, the President of the Association for that year, deliver his opening address, and I shall ever remember the gratification I felt, on hearing His Royal Highness make the following remarks:—"*Remembering that this Association is a popular Association, not a secret confraternity of men, jealously guarding the mysteries of their profession, but inviting the uninitiated, the public at large, to join them, having as one of its objects, to break down those imaginary and hurtful barriers, which exist between men of science and so-called men of practice.*" Surely, thought I at the moment of hearing these words, the friends who advised me to stay at home must have been mistaken. Little did I suppose, that before the end of the meeting, I should discover the practice of the Association to be so widely at variance with the theory of its constitution, as set forth in such flattering terms in the opening address of its Royal President.

The following morning I presented my Paper to the Committee of the Mathematical Section, and was told without any hesitation, I could not be permitted to read it; that the subject was prohibited from being introduced into the deliberations of the Association. For a short time, and with considerable earnestness, I endea-

voured to reason with the Committee as to the propriety of such a determination; but the Association had evidently arrived at the same conclusion as the late Baden Powell, viz., That "*no amount of attestation from innumerable and honest witnesses, would ever convince any one versed in mathematical science, that a person had squared the circle;*" and the attestation of so obscure and unknown an individual as the writer, in all probability produced in the minds of the Committee a feeling of the most profound contempt. Be this as it may, I reasoned in vain.

I then changed my tactics, and prepared a short Paper, entitling it, "*On the Relations of a Circle inscribed in a Square,*" and I again presented myself before the Committee, but with no better result. Subsequently, however, through the intervention of an individual member of the Committee, (J. Pope Hennessy, Esq., M.P. for King's County, Ireland, and to whom I take this opportunity of tendering my grateful acknowledgments,) my second Paper was inserted for reading in the programme of the following day, but was placed the last on a list of thirty Papers. All these Papers could not be disposed of that day, and many of them had to stand over till the next sitting of the Section, mine being among the number. What was my astonishment on finding it asserted in the programme of the following morning, that my Paper had been read the preceding day. This appeared to me to

be positive evidence of a determination to burke the subject, by any means, however dishonest, and I at once resolved upon my course. I took my seat in the Section, and waited until Sir William Rowan Hamilton, the Astronomer Royal of Ireland, took the chair for the day, in the absence of Lord Rosse. I then rose, and made my complaint, demanded to read my Paper, and gave the Section to understand, that I was not the man that would permit even the British Association to trifle with me. It was not an every-day scene in the Sections of the British Association, and Sir W. R. Hamilton will not have forgotten the circumstance. He permitted me to read the Paper. (See Appendix B.) Though short, it introduces to notice some of the fundamental truths of this important discovery.

I left a written copy of the Paper with the Secretary to the Section, and subsequently forwarded a printed one for insertion in the Transactions of the Association. If the reader will refer to the Report of the Association for 1859, he will find it recorded in the Transactions of the Mathematical Section, page 10, that J. Smith read a Paper "*On the relations of a Circle inscribed in a square.*" I should like to know how much wiser any reader of the Transactions of the Association would be, for this wonderful piece of information. Could this learned confraternity have devised any better method, of jealously guarding the mysteries of their profession? or, could they have af-

forded better evidence of the difference between the practice of the Association and the theory of its constitution, as enunciated by its Royal President?

During the course of the meeting, I accidentally met Mr. Airy, the Astronomer Royal of England, in the quadrangle of the College. I was not personally acquainted with him, but assuming that for the time being the members of the Association met on a footing of equality, I ventured to address him, respectfully asking him if he could afford me a few minutes' conversation on an important mathematical subject, stating that I believed I had made the discovery of the true Circumference and Area of a Circle. "*It would be a waste of time, Sir, to listen to anything you could have to say on such a subject,*" was his reply, and he attempted to change the conversation, by putting a frivolous question to me on an entirely different matter. I at once observed that I was considered guilty of an intrusion, begged Mr. Airy's pardon for it, and bid him "Good Morning."

On my return home, I commenced an enquiry into the relations of a Circle with a variety of other Geometrical figures, and very soon made some most important discoveries. I then felt justified in addressing the following Letter (with some slight verbal alterations) to Sir William Rowan Hamilton:

Barkeley House, Seaforth, near Liverpool,
14th February, 1860.

SIR WILLIAM,

This day twelve months, I published a small Pamphlet on "*The true Circumference and Area of a Circle*," and on the last day of sitting of the Mathematical Section of "The British Association for the Advancement of Science," at its meeting at Aberdeen, you presided in the absence of Lord Rosse, and you may remember, that after very considerable opposition and difficulty, I did succeed in reading a Paper, entitled, "*On the relations of a Circle inscribed in a Square.*" I subsequently sent you, and at the same time I sent to many others whom I thought likely to take some interest in the subject, a copy of the Paper I had read. (Duplicate enclosed.)

I have not received a single private communication in reference to it, and I am not aware it has ever been publicly noticed, except in one instance. The Editor of the *Athenæum*, did condescend to inform his readers, that I had read such a Paper at the meeting of the Association, in a paragraph of three lines, not more remarkable for their falsehood, than their absurdity.*

* "British Association, Section A. 'On the relations of a circle inscribed in a square.' The author enunciated a few well-known relations in imperfect decimal expression, derived from the approximate numerical expression for the circumference of a circle."—The *Athenæum*, 8th October, 1859.

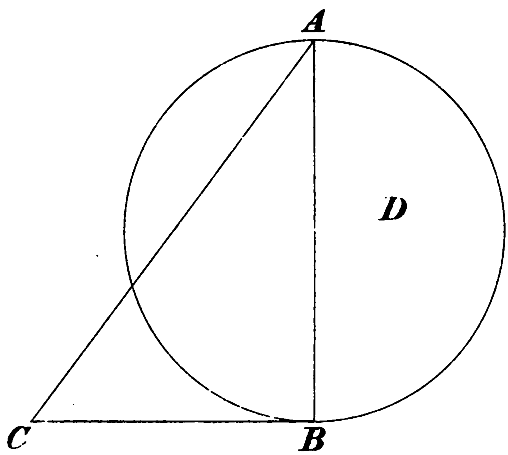
I spoke to your Royal Brother of England when in Aberdeen, who dismissed me most unceremoniously, giving me very plainly to understand, *that in his opinion, it would be a waste of time to listen to anything I could have to say on such a subject.* It is just possible, you may hold a similar opinion, but as it is my intention to publish on this interesting and important question, before doing so, I feel that it is due to you, having read my Paper in the Section at which you presided, again to direct your attention to the subject, and give you an opportunity of communicating with me, if you should think proper to do so, either admitting or denying the truth of the hypothesis I assume.

I have since the period referred to, examined the properties of a Circle, not only in its relation to a circumscribed and inscribed square, but also in its relation to other commensurable geometrical figures, and I send you herewith one instance, as an illustration of the truth of my theory. It is only one out of many I shall produce, all equally demonstrative.

I need not point out to you as a piece of information, that if the two sides of a right-angled triangle adjacent to the right angle, be in the ratio of 3 to 4, the triangle is commensurable, and the value of the third side, can be arithmetically given, either in whole numbers, or in perfect decimal expression.

But at this point, I beg to draw your attention to the

FIGURE A



fact, that this commensurable right-angled triangle, is a link in the chain of commensurable geometrical figures, connecting one with another in a most remarkable manner, so much so, that their connection may be reduced to a system, at once beautiful, harmonious, and thoroughly self-consistent.

In the following diagram, (see figure A,) let A B C be a commensurable right-angled triangle, of which the side B C is to the side A B, in the ratio of 3 to 4; and D, a circle, of which the diameter is equal to the perpendicular of the triangle.

In the first place, let the diameter of the circle, and perpendicular of the triangle, be 8.

On the orthodox hypothesis, $8 \times 3{\cdot}1416 = 25{\cdot}1328$, will be the approximative value of the circumference of the circle.

On my hypothesis, which, for the sake of distinction, I shall call the heterodox hypothesis, $8 \times 3{\cdot}125 = 25$, will be the exact circumference of the circle.

The perpendicular of the triangle is 8. Therefore, $\frac{3}{4}(8) = 6$, will be the base of the triangle. $\frac{5}{4}(8)$, or $\sqrt{(6^2 + 8^2)}$, $= 10$, will be the hypotheneuse of the triangle. And $6 + 8 + 10 = 24$, will be the value of the perimeter of the triangle.

Then, on the former hypothesis, the ratio between the circumference of the circle, and perimeter of the triangle, will be, as $25{\cdot}1328$ to 24. And on the latter hypothesis,

the ratio between the two, will be, as 25 to 24. And these ratios hold good, no matter how peculiar may be the decimal selected as the diameter of the circle and perpendicular of the triangle.

For example: Let the diameter of the circle, and perpendicular of the triangle, be 7·7.

On the orthodox hypothesis, $7 \cdot 7 \times 3 \cdot 1416 = 24 \cdot 19032$, will be the approximative value of the circumference of the circle.

On the heterodox hypothesis, $7 \cdot 7 \times 3 \cdot 125 = 24 \cdot 0625$, will be the exact value of the circumference of the circle.

The perpendicular of the triangle is 7·7. Therefore, $\frac{3}{4}(7 \cdot 7) = 5 \cdot 775$, will be the base of the triangle; $\frac{5}{4}(7 \cdot 7)$, or $\sqrt{(5 \cdot 775^2 + 7 \cdot 7^2)}$, $= 9 \cdot 625$, will be the hypotheneuse of the triangle; and $5 \cdot 775 + 7 \cdot 7 + 9 \cdot 625 = 23 \cdot 1$, will be the value of the perimeter of the triangle; and, as $25 \cdot 1328 : 24 :: 24 \cdot 19032 : 23 \cdot 1$; or, as $25 : 24 :: 24 \cdot 0625 : 23 \cdot 1$; and so far one hypothesis would appear to be just as good as the other.

But, reverse the operation, and let the circumference of the circle be the given quantity, say 60; and let it be required to find the values of the diameter of the circle, and the perimeter of the triangle.

On the heterodox hypothesis, $60 \div 3 \cdot 125 = 19 \cdot 2$, will be the exact value of the diameter of the circle, and perpendicular of the triangle; $\frac{3}{4}(19 \cdot 2) = 14 \cdot 4$, will be the base of the triangle; and $\frac{5}{4}(19 \cdot 2)$, or, $\sqrt{(14 \cdot 4^2 + 19 \cdot 2^2)}$, =

24, will be the hypotheneuse of the triangle; and, 19·2 + 14·4 + 24 = 57·6, will be the value of the perimeter of the triangle; and, as 25 : 24 : : 60 : 57·6 exactly; and on this hypothesis the ratio holds good, whether the circumference of the circle be the given quantity, to find the diameter of the circle and the perimeter of the triangle; or, the perpendicular of the triangle be the given quantity, to find the circumference of the circle.

On the orthodox hypothesis, 60 ÷ 3·1416 = 19·0985 &c. will be the approximative value of the diameter of the circle, and perpendicular of the triangle; $\frac{3}{4}$ (19·0985) = 14·323875, will be the base of the triangle; and $\frac{5}{4}$ (19·0985) = 23·873125, will be the hypotheneuse of the triangle; and, 19·0985 + 14·323875 + 23·873125 = 57·2955, will be the value of the perimeter of the triangle. But, these figures are less than those required by the orthodox ratio. For, as 25·1328 : 24 : : 60 : 57·2956, and it appears to me to be about as absurd to attempt to maintain the orthodox hypothesis, as it would be to maintain that 6 × 8 = 48, but that 8 × 6, is only equal to 47 and a fraction.

I have endeavoured to bring this subject under the notice of several gentlemen, who pride themselves on their high mathematical attainments, and judging from the reception I have met with at their hands, I am disposed to think that the philosophical world is not much wiser now, than it was in the days of Galileo, but,

" *Magna est veritas, et prevalebit,*"

and it is not in the power of ten thousand men of the highest genius, by their united efforts to annul this glorious fact.

Political, religious, and social freedom, have, however, made rapid strides even in our own day, in this privileged and happy nation, and it is now the right of a Briton, to be able to freely express an opinion, without incurring the risk of torture, or the danger of a prison; and in virtue of this privilege, I venture to address you thus freely, as a "great authority" on this subject, conceiving that the importance of it fully justifies me in doing so.

In conclusion, I may remark that my position in life is happily one of the most perfect independence. I have gone into this enquiry from a pure love of science, and a disinterested desire to promote it, and this course I shall continue to pursue. To myself personally, therefore, it is a matter of little consequence what course you may be pleased to adopt. I shall be glad however to find, that you do not consider the subject unworthy your attention, and in this respect form an honourable exception, among those of your professional brethren, with whom I have come into contact.

I remain, SIR WILLIAM,

Yours very respectfully,

JAMES SMITH.

Sir Wm. R. Hamilton, L.L.D., &c.,
Observatory, near Dublin.

P.S.—Having referred to an incident in connection with Mr. Airy, the Astronomer Royal of England, it might be said, if he were not made aware of it, that I was making a false charge, without having given him the opportunity of refuting it. To prevent this, I shall write him and enclose a copy of this letter, and if he should have changed his opinion, it will also give him the opportunity of admitting it.

I received from Sir W. R. Hamilton, by return of post, the following very courteous reply:

Observatory, near Dublin,
February 15th, 1860.

Sir,

I have received your letter, with its printed enclosure; another copy of this latter, (namely of the printed Paper,) had indeed reached me some months ago; but I did not understand that you required me to acknowledge it: nor would it have been a pleasant task to inform an ingenious gentleman, without necessity, of my entire disagreement from his views.

But since, while reminding me of my having had the honour to preside in the absence of Lord Rosse, when you read your paper in Section A of the British Association, at Aberdeen, last year, you are pleased "again to direct my attention to the subject, and to give me an opportunity of communicating with you, if I should think

proper to do so, admitting or denying the truth of the hypothesis you assume," in extracting which passage from your letter, I merely make such substitutions as that of me for you, &c. I suppose that you might be more displeased by my remaining silent, than by the distinct expression of my dissent.

I understand you to maintain, that if the diameter of a circle be represented by the number 8, the circumference of the same circle will then be represented, without any error, by the number 25. Now, Sir, I do not expect you to attach the slightest weight to any opinion of *my own*, on this or any other subject. But it will much surprise me, if you shall not find Mathematicians *unanimous* in their rejection of that result. That 8 circumferences of a circle *exceed* 25 diameters, is (I conceive) a theorem as completely certain and established in mathematics, as that the three angles of a plane triangle are together equal to two right angles.

When you next publish, if you shall think it needful to mention the fact of my having been in the chair of the Section, I hope that you will be so good as to state, at the same time, that my opinion, or rather conviction upon the subject, is entirely opposed to your own.

I have the honour to be, Sir,
Your obedient servant,
WM. ROWAN HAMILTON.

James Smith, Esq.,
Barkeley House, Seaforth, near Liverpool.

I addressed the following note to Mr. Airy, enclosing copy of my letter to Sir W. R. Hamilton :—

Barkeley House, Seaforth,
Liverpool, 14th Feby., 1860.

Sir,

I herewith enclose you a copy of a letter I have this day addressed to Sir Wm. R. Hamilton.

As I have referred in it to a circumstance in which you are personally concerned, I consider it a matter of courtesy to make you acquainted with all I have said respecting you.

I am, Sir,
Yours very respectfully,
JAMES SMITH.

G. B. Airy, Esq., F.R.S., &c.,
Royal Observatory, Greenwich.

The following was Mr. Airy's reply :—

Royal Observatory, Greenwich,
15th February, 1860.

Sir,

I have this day received your letter of the 14th Inst., enclosing copy of one addressed to Sir W. R. Hamilton.

As regards Sir W. R. Hamilton, I have no remark to offer. As regards myself, you will doubtless remark, that

every person has a right to publish his own views, by any inoffensive method which he may think best, but that this gives him no command, as by-right, of the most valuable possession of other persons, namely, their time.

I am, Sir

Your obedient servant,

G. B. AIRY.

James Smith, Esq.

Such are some of the obstacles I have had to contend against, in announcing the discovery of a glorious scientific truth; and such, for some time to come, may probably be the fate of others, whose views, however true and important, may happen to run in antagonism to the prejudices of the scientific world. But truth, spring from whence it may, will ultimately triumph in spite of all opposition. I am not a young man, and may not live to see it, but the day will arrive, when truth by her own inherent powers shall rise above the horizon, and shining forth in all her brightness, shall dispel the darkness which now reigns on this and kindred subjects; and shall, on the one hand, "*reduce*" to their natural and proper level, that "learned fraternity" who at present contrive, like the astrologers of old, to make a mystery of their craft, and to jealously guard it; and shall, on the other hand, "*induce*" thousands of persons of both sexes to embrace the study of the sublime and glorious works of our Creator, as manifested in the

wonderful evolutions of the heavenly bodies, who at present are intimidated into the belief, that the study of Astronomy is a something utterly beyond the capacity of the average intellect of mankind.

I may, on a future occasion, direct public attention to the importance of this discovery, in its practical application to Astronomical, Nautical, and Mechanical Science.

JAMES SMITH.

Barkeley House, Seaforth,
Liverpool, 1st April, 1861.

CORRESPONDENCE.

EMINENT MATHEMATICIAN, to the AUTHOR of "The Question:—Are there any commensurable relations between a Circle and other Geometrical Figures?"

London, July 14th, 1860.

SIR,

You have done me the honour to enclose to me a copy of your pamphlet on the Squaring and Rectification of the Circle.

If you will allow me to address you by name, I will in the space of one page of writing, point out that your highly ingenious reasoning rests on a fallacy.

I am, Sir,

Obediently yours,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 17th July, 1860.

SIR,

Your letter of the 14th instant only came into my hands this morning.

My only object is to arrive at the truth on the interesting question of "The Quadrature of the Circle;" and if there really be a fallacy in my reasoning, I shall be glad to be convinced of it.

I beg to enclose a copy of the paper read by me, in the Mathematical Section of "The British Association for the Advancement of Science," at its meeting at Aberdeen last year.

I am, Sir,

Yours obediently,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, July 18th, 1860.

SIR,

The fallacy which pervades your argument in the paper respecting the Squaring of the Circle, is, that you treat linear and square units as if they were identical. It is very true, abstractedly, that one, and the square of one, are expressed by the same symbol, but they are very different things. As for instance, if you have an estate bounded by a fence one mile long on each side of a square, it is true to say that the side of the square is 1, it is also true to say that the area is 1, but you speak of 1 *linear* unit as to the fence, and 1 *square* unit as to the estate, and it would be absurd to say that the fence of one side of the estate is equal to the estate itself; equally absurd is it to say that the *side* of any square is equal to its *area*, or any area.

To apply this remark to your paper, take the words,

"It is admitted that the circumference of the Circle B is equal to four times its area, and the side of Square C being equal in numerical value to the area of Circle B, the circumference of Circle B is also equal in value to four times the side of Square C." (See Appendix B.)

But no such thing can be admitted. All that can be admitted is, that the *number* of square units in one case corresponds with the *number* of linear units in the other. You might as well say that 50 pounds is equal to 50 thousand pounds, because the figure 50 is used in each case. The unit is a pound in one case, and a thousand pounds in the other.

But there is another fallacy further down in the same paper, where it is said, "I may here remark, that if any *other number* than ·7854 be assumed as the area of a Circle of which the diameter is unity, the only effect of it would be to change the values of the circumference of Circle B, and the side and area of Square C, but would not, in any respect whatever, affect the principle involved in the ratios."

The *area* of a *Circle* whose diameter is linear unity, is fixed, and cannot have more than one value, measured in square units. If that value be ·7854, *no "other number"* can be assumed as representing it. And when you assume the ratio of 1 to ·78125, you must abandon the hypothesis that 1 to ·7854 represents the true ratio.

In point of fact, if you take the ratio 1 to ·78125 the curve is not a circle, but an inscribed polygon, and one which is quadrable.

Any doubt which you may have as to the real ratio of diameter to circumference in a circle, is readily removed by any treatise on Trigonometry, where the method of finding the length of the circular measure of any part, or the

whole, of a circle, is demonstrated beyond the possibility of cavil. I need hardly add that a fixed ratio cannot be *both* 1 to ·78125, and also 1 to ·7854. If you confine your reasoning to the simple case, "What is the ratio of the diameter to circumference," omitting all reference to the subject of areas, I think you will not fail to arrive at the full understanding of the matter.

I am, Sir,

Yours obediently,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 24th July, 1860.

SIR,

The general tenor of your letter of the 18th instant would seem to imply, that you understand me to hold the opinion, that the area of a circle of which the diameter is linear unity, is not a fixed quantity; it is therefore necessary, before proceeding to deal with your arguments, that I should lay down, in as distinct terms as possible, certain propositions which shall explicitly embody my views on the interesting question of "The Quadrature of the Circle."

I affirm, that there is a relation between the diameter and circumference of a circle, that this relation is definite and commensurable, and can be expressed in figures with perfect exactness.

I affirm, that the circumference and area of a circle of which the diameter is linear unity, are definite and

commensurable quantities, and can be expressed in figures without the least error.

I affirm, that the relation between the diameter and circumference of any circle may be expressed in the following terms:—For every linear unit contained in the diameter, there are three and one-eighth linear units contained in the circumference.

And lastly, I affirm, that the figures 3·125, represent the circumference, and ·78125 the area of a circle, of which the diameter is linear unity, and that both are fixed and unchangeable quantities.

In your letter you make the following statement:— "The area of a circle whose diameter is linear unity is fixed, and cannot have more than one value, measured in square units. If that value be ·7854, *no 'other number'* can be assumed as representing it." You are no doubt aware that on the commonly accepted data, the figures 3·1415926, &c., would represent a nearer approximation to the circumference, and ·78539815, &c., a nearer approximation, measured in square units, to the area of a circle, of which the diameter is linear unity, than 3·1416, and ·7854, the figures which for practical purposes are usually adopted as sufficiently accurate.

If the area of a circle of which the diameter is linear unity be fixed, and cannot have more than one value, (and I admit the fact), and if you maintain ·7854 to be that value, you must abandon the orthodox theory. If, on the other hand, you profess to accept the orthodox theory, you must abandon the figures ·7854 as the fixed and unchangeable value of the area of a circle of which the diameter is linear unity.

If you venture to say the circumference and area of a circle of which the diameter is linear unity, have a fixed

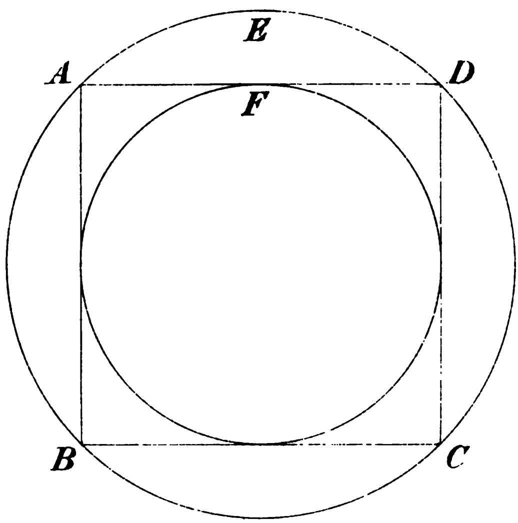
value, but that this value cannot be expressed exactly in figures; this I deny, and maintain, on the contrary, that 3·125 exactly represents the one, and ·78125 exactly represents the other; and I cannot discover any thing in your letter, at all calculated to throw a doubt into my mind as to the truth of my figures, much less to convince me that I am in error. You will of course now understand, that I maintain, as the only true ratio of diameter to area in a circle, of which the diameter is linear unity, that of 1 to ·78125, and abandon all others.

You are of opinion that the fallacy which pervades my argument, is that of treating linear and square units as if they were identical. I really cannot plead guilty to this charge. You illustrate your statement by supposing an estate in the form of a square, and containing an area of one square mile. The fence on each side of the estate will of course be one linear mile. You say, "It is very true, abstractedly, that 1, and the square of 1, are expressed by the same symbol, but they are very different things."

Now, symbols may be either arithmetical or algebraical, the notation of the former is figures, the notation of the latter is letters, and by an application of the two, to the consideration of the question at issue, I think the fallacy of your reasoning may be made apparent.

Let the fence on one side of the estate be represented by the algebraical symbol A, and let the area of the estate be represented by the algebraical symbol B, and let the value of the algebraical symbol A, be represented by the arithmetical symbol 1. It may be one yard, one furlong, one mile, or one any thing. It would be very absurd if I were to say that A and B were the same thing, but it would be equally absurd, if I were to attempt to express the value of B, by any other arithmetical symbol

FIGURE I.



than 1, until I am informed whether the algebraical symbol A, stands for one yard, one furlong, one mile, or one something else, and then it ceases to be a geometrical, and is resolved into an arithmetrical problem.

Let D represent the diameter, C the circumference, and A the area of a circle, and let the value of D be represented by the arithmetical symbol 4.

Then, on the orthodox data, $4 \times 3{\cdot}1416 = 12{\cdot}5664$, will be the arithmetical value of C; $\frac{1}{2}$ (C) $\times$ $\frac{1}{2}$ (D), $= \frac{1}{2}(12{\cdot}5664) \times \frac{1}{2}$ (4), $= 6{\cdot}2832 \times 2, = 12{\cdot}5664$, will be the arithmetical value of A. Or, on the writer's hypothesis, $4 \times 3{\cdot}125 = 12{\cdot}5$, will be the arithmetical value of C; $\frac{1}{2}$ (C) $\times$ $\frac{1}{2}$ (D), $= \frac{1}{2}$ (12·5) $\times$ $\frac{1}{2}$ (4), $= 6{\cdot}25 \times 2, = 12{\cdot}5$, will be the arithmetical value of A. It would be very absurd if I were to say, that C and A represented the same thing, but it would be equally absurd, if I were to attempt to express the values of C and A, by any other than the same arithmetical symbols, as ascertained either on one hypothesis or the other.

In the enclosed diagram (see Fig. I.) let A B C D be a square, E a circle circumscribed about it, and F a circle inscribed in it, the area of which is admitted to be equal in arithmetical value to half the area of circle E. Let X represent the circumference of circle E, Y the area of circle E, and Z the area of circle F, and let the diameter of circle E, be represented by the arithmetical symbol 8.

Then, on the orthodox data, $8 \times 3{\cdot}1416 = 25{\cdot}1328$, will be the arithmetical value of X; $\frac{1}{2}$ (X) $\times$ $\frac{1}{2}$ (8), $= \frac{1}{2}$ (25·1328) $\times$ $\frac{1}{2}$ (8), $= 12{\cdot}5664 \times 4, = 50{\cdot}2656$, will be the arithmetical value of Y; $\frac{1}{2}$ (Y), $= \frac{1}{2}$ (50·2656), $= 25{\cdot}1328$, will be the arithmetical value of Z. Or, on the writer's hypothesis, $8 \times 3{\cdot}125 = 25$, will be the arithmetical value of X; $\frac{1}{2}$ (X) $\times$ $\frac{1}{2}$ (8), $= \frac{1}{2}$ (25) $\times$ $\frac{1}{2}$ (8), $= 12{\cdot}5 \times 4, = 50$,

will be the arithmetical value of Y ; $\frac{1}{2}$ (Y), $= \frac{1}{2}$ (50), $= 25$, will be the arithmetical value of Z. Again, it would no doubt be very absurd, if I were to say that X, which represents the circumference of circle E, and Z, which represents the area of circle F, were the same thing ; but it would be equally absurd, if I were to attempt to express the values of X and Z, by any other than the same arithmetical symbols, as ascertained either on one hypothesis or the other.

In the first part of the paper read by me at "The British Association for the Advancement of Science," I have simply directed attention to the fact, that whatever hypothetical data be taken to represent the area of a circle, of which the diameter is linear unity, whether ·7854 or any other number—I might say ·785398I5, or ·78125—the ratio that exists between the area of any square and the area of a circle inscribed in it, exists likewise between the numerical value of the four sides of such square, and the circumference of the inscribed circle ; between the area of the circle and the area of a square, of which the four sides are together equal in numerical value to the circumference of the circle ; and between the side of the square containing the inscribed circle, and the side of a square of which the numerical value is equal to one-fourth of the circumference of the circle ; and for this purpose the paragraph to which you refer is superfluous, and might have been omitted altogether. But suppose the paragraph to have run thus :—" It cannot be denied that the arithmetical symbols which represent the circumference of circle B, are equal in numerical value to four times the arithmetical symbols which represent its area,"—who could have disputed the fact, and where is the fallacy in the reasoning ?

The paragraph in your letter contrasting 50 with 50 *thousand* I pass by, for 50 thousand is neither an arithmetical nor an algebraical symbol.

In another paragraph you say,—"In point of fact, if you take the ratio 1 to ·78125, the curve is not a circle but an inscribed polygon, and one which is quadrable." This is mere assertion, without any attempt at proof; and in reply I might say,—If you take the ratio 1 to ·7854, the curve is not a circle, but a circumscribed polygon; but this is merely to bandy words, and can be of no service to either of us in the solution of this question.

You appear to think that a reference to any treatise on Trigonometry, would remove all doubt as to the real ratio of diameter to circumference in a circle. As the accuracy of the circular measurement of an angle depends on the accuracy of the circumference of a circle, it appears to me, that while the commonly accepted data for the circumference of a circle, is admitted to be only an approximative value, so long must we be content with an approximation to the true value of the circular measure of an angle, and therefore, that the science of Trigonometry cannot be made available for the solution of this question.

Neither in the first part of my paper, nor in my pamphlet as far as page 8, have I done more than give my reasons for rejecting the orthodox data, which is a minor proposition. My major proposition I have attempted to establish from page 8 to the end of my pamphlet. All your remarks have reference to the former, the latter you have left untouched.

In conclusion, I beg to thank you for the trouble you have taken, but respectfully submit that you have entirely failed to disprove my proposition:—"That for every

linear unit contained in the diameter of a circle, there are three and one eighth linear units contained in the circumference.

I am, Sir,

Yours obediently,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, July 26th, 1860.

SIR,

I will endeavour to put the "question" in a more simple form, so as to obviate the liability, to mistake what is meant by the words used.

First. The ratio of diameter to circumference in a circle, is a fixed, invariable, ratio.

Second. The terms of a ratio may be multiplied or divided by a number, without altering the ratio itself, but one term alone of the ratio cannot be altered without altering the ratio.

Third. If the ratio of diameter to circumference in a circle, be 1 to 3·14159 &c., it cannot be 1 to 3·125. And if the said ratio be 1 to 3·125, it cannot be 1 to 3·14159 &c.

I have no time (to save the post), and will therefore postpone the proof of the ratio 1 to 3·14159 &c., until to-morrow, in the meantime asking you to furnish me with a proof, (not an assumption) that the true ratio is 1 to 3·125.

To prevent the ambiguity of linear and square units,

let us confine our proof, in the first instance, strictly to the ratio of line to line, namely, diameter to circumference.

I am, Sir,

Yours truly,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 27th July, 1860.

SIR,

I have the honour to acknowledge the receipt of your favour of yesterday's date.

In reply, I beg to say I have never denied any one of the three propositions you lay down; on the contrary, I maintain that every one of them is strictly true, as expressed in your own phraseology.

You are of opinion, that the ratio of diameter to circumference in a circle is, as 1 to 3·14159 &c. I say the ratio between the two is, as 1 to 3·125. We are agreed that both cannot be true, and the point at issue between us is, which of the two is correct?

You ask me for a proof (not an assumption) that 1 to 3·125 is the true ratio.

In my pamphlet I have given what I consider several proofs, and I may here mention that they are such as are deemed satisfactory by several mathematicians, who have already expressed their conviction that I am right, and that 1 to 3·125, and no other, is the true ratio of diameter to circumference in a circle. The proofs of the fact are, however, legion, and I select the following, which are

strictly confined to the ratio of line to line, but embracing something more than the mere ratio of diameter to circumference.

In the enclosed diagram (see Fig. II.) let A be a circle, B C D a right-angled triangle, of which the side B C is the diameter of the circle A, and let the value of the line B C be 1.

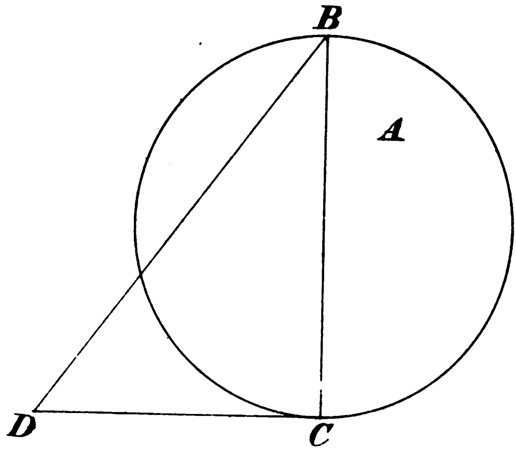
I say the value of the line representing the circumference of the circle is 3·125, and the ratio between the diameter and circumference 1 to 3·125.

Let the value of the side D C of the triangle B C D be ·75, the lines D C and B C represent the two sides of a right-angled triangle adjacent to the right angle, and are to each other in the ratio of 3 to 4. For, as 3 : 4 : : ·75 : 1 : and $DC^2 + BC^2, = {·75}^2 + 1^2, = ·5625 + 1, = 1·5625$; and $\sqrt{1·5625} = 1·25$, must be the value of B D, the hypotheneuse, or third side of the triangle B C D, and the triangle B C D, is a commensurable right-angled triangle, of which the arithmetical symbols ·75, 1, and 1·25, represent the values of the three sides respectively with perfect exactness, and the ratio between the perimeter of the triangle and circumference of the circle, is, as 24 to 25. For, $·75 + 1 + 1·25 = 3$ is the perimeter of the triangle, and, as 24 : 25 : : 3 : 3·125 the circumference of the circle.

If the terms of the ratio be multiplied by 8, then, $1 \times 8 = 8$, and $3·125 \times 8 = 25$, and the ratio of the diameter to the circumference of the circle will be as 8 to 25, and is not altered from the ratio of 1 to 3·125.

Let the line B C in the diagram, which represents the diameter of the circle, and also one side of the triangle B C D, be 8. The side D C is to the side B C of the triangle, in the ratio of 3 to 4 by construction. Therefore, $\frac{3}{4}(8) = 6$ will be the value of the side D C ; 8 is the

FIGURE II.



given value of the side B C ; and $6^2 + 8^2, = 36 + 64, = 100$; and $\sqrt{100} = 10$, must be the value of B D, the third side of the triangle B C D ; and $6 + 8 + 10 = 24$, will be the value of the perimeter of the triangle, and the ratio between the perimeter of the triangle and circumference of the circle, is, as 24 to 25, as in the previous example.

The terms of the ratio may be multiplied by the most peculiar decimal quantity, and the demonstration will still be perfect.

For example, $1 \times 7{\cdot}1 = 7{\cdot}1$, and $3{\cdot}125 \times 7{\cdot}1 = 22{\cdot}1875$, and the ratio of the diameter to the circumference of the circle will be, as 7·1 to 22·1875, and is not altered from the ratio of 1 to 3·125. Then, $\frac{3}{4}(7{\cdot}1), = 5{\cdot}325$, will be the value of the side D C ; 7·1 will be the value of the side B C ; and $5{\cdot}325^2 + 7{\cdot}1^2, = 28{\cdot}355625 + 50{\cdot}41, = 78{\cdot}765625$; and $\sqrt{78{\cdot}765625} = 8{\cdot}875$, will be the value of B D, the third side of the triangle B C D ; and, $5{\cdot}325 + 7{\cdot}1 + 8{\cdot}875 = 21{\cdot}3$, will be the value of the perimeter of the triangle B C D; and as $24 : 25 :: 21{\cdot}3 : 22{\cdot}1875$ the circumference of the circle.

The circumference of the circle may be the given quantity, say 60.

Then, $60{\cdot} \div 3{\cdot}125 = 19{\cdot}2$, will be the diameter of the circle, and the ratio of the diameter to the circumference of the circle, will be as 19·2 to 60, and is not altered from the ratio of 1 to 3·125.

Then $\frac{3}{4}(19{\cdot}2) = 14{\cdot}4$, will be the value of the side D C ; 19·2 will be the value of the side B C ; and $14{\cdot}4^2 + 19{\cdot}2^2, = 207{\cdot}36 + 368{\cdot}64, = 576$; and $\sqrt{576} = 24$, will be the value of B D, the third side of the triangle B C D; and the ratio between the perimeter of the triangle and circumference of the circle is, as 24 to 25, as in the previous examples. For, $14{\cdot}4 + 19{\cdot}2 + 24 = 57{\cdot}6$, and as

24 : 25 : : 57·6 : 60, the given value of the circumference of the circle.

And now, will you permit me to ask you to inform me, by means of your data, what is the diameter of a circle, if the circumference of it be 60?

In conclusion, I may give you the following extract from a letter received by me three days ago. The writer is a known mathematician and astronomer:—"Permit me to congratulate you on the very perfect way in which you have solved the great problem. This is another instance of the folly of trusting to mere 'authority,' instead of insisting on evidence being produced."

I am, Sir,

Yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, 27th July, 1860.

SIR,

I enclose two different proofs of the question, regarding the ratio of the circumference to diameter in a circle.

I have purposely avoided naming the area, because I well know that it is with respect to the area that so much confusion of idea has existed.

In the old "Hindu" treatise called "Leelawutter," the rule given is, Multiply the diameter by 3927, and divide the product by 1250. And Archimedes' rule was, Multiply by 22 and divide by 7. Both these calculations are of course rude ones.

I think you will find it quite impossible to shew that the ratio of diameter to circumference is 1 to 3·125. Look carefully over your argument, and you will see that you have imported the ratios of an area too soon. First establish the ratio of line to line, and then apply that ratio in finding the area.

If you like to describe a large circle with a Gunter's chain on the flat sands near Liverpool, you will easily satisfy yourself by measurement, that all the world is not in the wrong, and yourself only in the right.

I am, Sir,

Obediently yours,

E. M.

PROOFS.

To find the ratio of diameter to circumference of a circle, or rather to find an approximation to such ratio, which may be made as close as we desire—

Let T denote the tangent of an arc A, the radius being supposed equal to 1, then the length of the arc A is, by the well known infinite series, equal to

$$T - \tfrac{1}{3}T^3 + \tfrac{1}{5}T^5 - \tfrac{1}{7}T^7 + \tfrac{1}{9}T^9 - \&c. \quad (1.)$$

Suppose the arc we wish to measure is one of 30°, which is $\frac{1}{12}$ the whole circumference. Its tangent being $\frac{1}{\sqrt{3}}$, let us write $\frac{1}{\sqrt{3}}$ for T in the above series, which now becomes

$$\frac{1}{\sqrt{3}} \times \left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \frac{1}{9.3^4} - \&c. \right. \quad (2.)$$

Multiply this by 12, and you will obtain the whole length of the circumference, since $12 \times 30° = 360°$.

In figures this works out as 3·141592653589793 &c.

Again, suppose the arc is one of 18°.

The tangent of 18° is $\sqrt{1-2\sqrt{\frac{1}{5}}}$, which may be substituted in series (1) and leads to the same result. Or

again, take the arc $22\frac{1}{2}°$. Its tangent is $2 - \sqrt{3}$. Substitute $2 - \sqrt{3}$ for T in series (1), and the same figures are obtained as before.

As, however, this series (1) may not be familiar to you, I subjoin another and entirely different proof.

The radius of a circle being 1, the sines, tangents, &c., for every minute in its circumference have been calculated and published in well known tables.

Now, the sine of 1″, or the $\frac{1}{21600}$ part of the circumference, is ·000290888208664... The tangent of 1″ or the $\frac{1}{21600}$ part of the circumference is ·000290888208668 ... Multiply both these quantities by 21600 to get the entire circumference, and the result is 3·1415926536161 ... and 3·1415926535767 ...

Now, all the sines together make up the inscribed polygon, and all the tangents together make up the circumscribed polygon. And the circle's circumference manifestly lies between the inscribed and circumscribed polygon of 10800 sides, therefore its length lies between the numbers above given, right to 9 decimals, therefore the ratio of radius to semi-circumference is 1 to 3·141592653.... a series which has been calculated to 209 places of decimals without terminating.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, 28th July, 1860.

SIR,

I believe I state your argument fairly as follows:—

Let A B C (see Fig. III.) be a rational right-angled triangle, whose perimeter = 3 .. That is, let A C = 1, B C = $\frac{3}{4}$

FIGURE III.

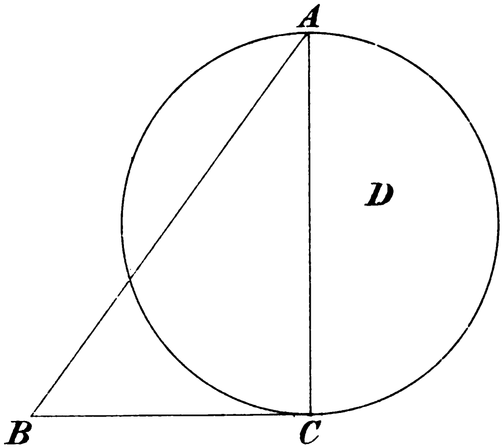
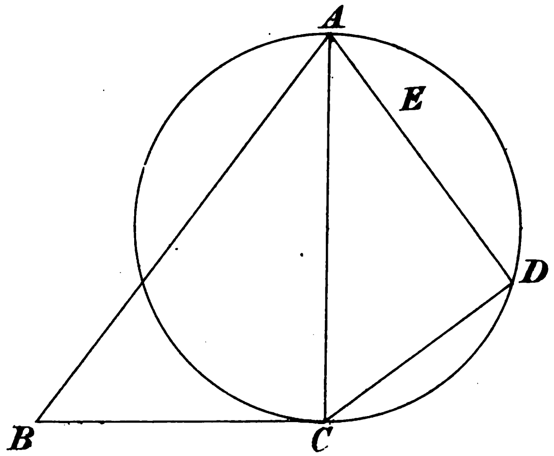


FIGURE IV



A B = $1\frac{1}{4}$; On A C as diameter, describe the circle D, which by hypothesis shall be equal to three and one-eighth times the length of A C.

Then the perimeter of the triangle will be to the perimeter of the circle, as 24 to 25.

The remainder of your letter consists of examples, the truth of which follow, as a matter of course, if the above reasoning is correct, but I beg, before remarking on the subject further, to ask whether I have rightly stated your argument.

I am, Sir,

Yours obediently,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 30th July, 1860.

SIR,

In your letter of the 28th inst., you have stated my argument with perfect accuracy.

The following information may be of assistance to you, in the consideration of my letter of the 27th inst., in regard to which the question you ask has reference.

In the enclosed diagram (see Fig. IV.) let A B C be a rational right-angled triangle, of which the sides C B, A C, adjacent to the right angle, are in the ratio of 3 to 4; and A D C a rational right-angled triangle, of which the sides D C, A D, adjacent to the right angle, are also in the ratio of 3 to 4; With A C as diameter, describe the Circle E. Let the value of A C be represented by the

arithmetical symbol 1, and, by hypothesis, let the circumference of a circle be three and one-eighth times its diameter.

Then, in the triangle A C B, C B = ·75, A C = 1, A B = 1·25; in the triangle A D C, D C = ·6, A D = ·8, A C = 1; and D C is proportional to C B, A D to A C, A C to A B, and the perimeter of the triangle A D C, to the perimeter of the triangle A C B, and the ratios are in every case as 4 to 5. For, as 4 : 5 : : ·6 : ·75; as 4 : 5 : : ·8 : 1; as 4 : 5 : : 1 : 1·25. The value of the perimeter of the triangle A D C is, $·6 + ·8 + 1 = 2·4$. The value of the perimeter of the triangle A C B is, $·75 + 1 + 1·25 = 3$; and as 4 : 5 : : 2·4 : 3.

The circumference of the circle is by hypothesis 3·125. The perimeter of the triangle A C B is by construction equal to 3. And the perimeter of the triangle A C B is to the circumference of the circle in the ratio of 24 to 25. For as 24 : 25 : : 3 : 3·125.

Then, if 24 be multiplied by 4, and 25 be multiplied by 5, (4 and 5 being the numbers representing the ratio of the proportionals in the lines of the triangles,) the products are 96 and 125, and these figures represent the ratio between the perimeter of the triangle A D C, and the circumference of the Circle E. For, as 96 : 125 : : 2·4 : 3·125.

Again, the circumference of the circle may be the given quantity, say 60.

Then, $60 \div 3·125 = 19·2$, will be the value of the side A C; $\frac{5}{4}(19·2) = 24$, will be the value of the side A B; and $\frac{3}{4}(19·2) = 14·4$, will be the value of the side B C, in the triangle A B C; and $19·2 + 24 + 14·4 = 57·6$, will be the value of the perimeter of the triangle A B C; and the ratio between the perimeter of the triangle A B C, and circumference of the Circle E, is, as 24 to 25. For as

24 : 25 : : 57·6 : 60, the given value of the circumference of the circle.

Again, the value of A C the hypotheneuse of the triangle A D C is 19·2. $\frac{4}{5}$ (19·2) = 15·36, will be the value of the side A D; $\frac{3}{5}$ (19·2) = 11·52, will be the value of the side D C, in the triangle A D C; and 19·2 + 15·36 + 11·52 = 46·08, will be the value of the perimeter of the triangle A D C; and the ratio between the perimeter of the triangle A D C, and circumference of the Circle E, is, as 96 to 125. For as 96 : 125 : : 46·08 : 60, the given value of the circumference of the circle.

In this diagram we have one geometrical figure entirely within the circle, and another partly within and partly out of it, and the line A C is common to all the three figures, and on the writer's hypothesis, there are commensurable relations not only between the perimeters of the triangles, and the circumference of the circle, but also between triangle and triangle, and a commensurable relation is established between any one of these geometrical figures, and either of the other two which cannot be dissolved.

No other arithmetical symbols can possibly produce the same result, and I would ask you, Is it possible to resist the inference, that the ratio of diameter to circumference in a circle is, as 1 to 3·125 ?*

I may here mention that there are commensurable relations between a circle and an inscribed equilateral triangle, between a circle and an inscribed hexagon, and between a circle and many other geometrical figures,

* It may be observed, that if the side B C of the triangle A C B, and the side A D of the triangle A D C, be produced, they will meet at a point, say E, describing a right-angled triangle C D E, of which the two sides adjacent to the right angle will be in the ratio of 3 to 4, and the triangle C D E, together with the triangle A D C, will be equal to the triangle A C B.

which, on the writer's hypothesis, are all equally beautiful, consistent, and harmonious.

Your letter of the 27th inst. would seem to imply, that the two different proofs which accompany it, regarding the ratio of circumference to diameter in a circle, are, in your opinion, unanswerable, must necessarily end the controversy between us, and justify the touch of sarcasm with which the letter concludes.

It is not exactly so; I assure you I can, when it suits my convenience, point out, without the least difficulty, the fallacy which pervades the demonstration in both instances.

I am, Sir, yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, August 3rd, 1860.

SIR,

As I have stated your argument accurately, I can now have no difficulty whatever in pointing out the fallacy which pervades it.

All that you say as to the triangles is right, but when you assume that on a diameter, one, a circle can be drawn, whose circumference is exactly equal to $3\frac{1}{8}$, you are taking that for granted *which has to be proved*, and moreover which is known by every one who has read the most elementary treatise on Trigonometry, *to be untrue*. If such a circle as you suppose could be drawn, that is, if the ratio 1 to $3\frac{1}{8}$ could exist, it would exist in all circles, and *no* circle could be drawn unless to that ratio.

There is no wonder that you find your circle and

your triangles agree, because you have arbitrarily assumed that you know the ratio between the diameter and circumference, which, if it be known, at once enables you to describe figures agreeing with the circle. Suppose I assume the ratio as hypothetically equal to 1 to 3. I then proceed to show that the perimeter of the supposed rational triangle is as you have rightly shown equal to 3. But now, should I be correct in arguing that the perimeter of the triangle and that of the circle were equal?

No. You or any one would say, Prove your major premiss; shew that the ratio of diameter to circumference is 1 to 3, and then I will admit your proposition, but not otherwise.

So say I; shew that such a circle can exist as one which has a ratio 1 to $3\frac{1}{8}$, and I will admit that a rectilinear figure equal to it can be described.

Remember that the well-known figures 3·141592 &c. have been tested by actual measurement not only of a degree of the meridian, but by General Roy, in his survey of Great Britain. Take a chain, and a radius of 100 feet, and stake out a semi-circle, you may do the same thing on a smaller scale as he did upon a larger one. But small as the scale is, you will find a string stretched along the stakes too short by an appreciable and convincing gap, if you adopt the hypothesis of 1 to $3\frac{1}{8}$ exactly.

You wished me to say what would be the diameter of a circle whose circumference is 60. The answer is, that it would be 19 and a fraction, which being as far as has been calculated, a decimal of 209 places of figures, cannot very easily be ascertained, but it would begin ·099 &c.

Let me conclude by observing that I meant no sarcasm in my letter. I recognize you as a bold inquirer after truth, not deterred by the authoritative dictation of

others, and as such inquirer I do earnestly beg you to buy a treatise on Trigonometry, and see the proof which all such books supply, of what is called Circular Measure, and study the reasoning on which the tables which guide every ship across the ocean are constructed, every one of which, if your hypothesis be correct, is erroneous.

I am, Sir,

Obediently yours,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 6th August, 1860.

SIR,

I have the honour to acknowledge the receipt of your favour of the 3rd inst., which in due course should have come to hand on the 4th, but which from some cause has only just reached me.

In one paragraph of your letter you state:—"All you say as to the triangles is right." In another paragraph you say:—"There is no wonder that you find your circle and your triangles agree, because you have arbitrarily assumed that you know the ratio between the diameter and circumference, which if it be known, at once enables you to describe figures agreeing with the circle." In a third paragraph you say:—"Shew that such a circle can exist as one which has a ratio 1 to 3·125, and I will admit that a rectilinear figure equal to it can be described."

Taking all these paragraphs in connection, I believe I fairly state your meaning as follows:—

Assuming (whether rightly or wrongly) the ratio of diameter to circumference in a circle to be 1 to 3·125,

then, with reference to the diagram in my letter of the 30th ult., and taking this arbitrarily assumed data, you admit; That the perimeter of the triangle A C B, is to the circumference of the Circle E, in the ratio of 24 to 25; That the perimeter of the triangle A D C, is proportional to the perimeter of the triangle A C B, and the ratio between them as 4 to 5; That the perimeter of the triangle A D C, is to the circumference of the Circle E, in the ratio of 96 to 125; And that this ratio is obtained by multiplying ($24 \times 4 = 96$ and $25 \times 5 = 125$), the figures which represent the ratio between the perimeter of the triangle A C B, and the circumference of the Circle E, by the figures which represent the ratio between the perimeter of the triangle A D C, and the perimeter of the triangle A C B.

Before proceeding further with the argument contained in your letter, I beg to ask whether I have correctly understood your meaning.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

Isle of Wight, 8th August, 1860.

SIR,

Your letter of the 6th has been forwarded to me here. In reply I beg to state that you correctly describe my comment on your argument, in respect of the perimeter of the triangle A B C, to the perimeter of the circle; namely, 24 to 25. I said nothing, I think,

about the triangle A D C, but that need not affect the argument, because my comment is simply that you cannot assume such a circle as that which you have assumed; you must begin by proving it can have an existence.

Of course the triangles may be drawn so as to have any ratio *to each other* which you may select. But if you desire to assign a ratio between *circle* and triangle, you must first show the ratio between circumference and diameter.

If there be such a circle as that which you assume, the whole problem is solved, for the ratio being finite, rectilinear figures only require the application of common arithmetic for their description.

I am, Sir,

Yours obediently,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 11th Aug., 1860.

SIR,

I have the honour to acknowledge the receipt of your favour of the 8th Inst.

Before proceeding to argue the question further, permit me to draw your attention to the stages through which we have passed, and the point at which we have arrived, in the controversy between us, on the interesting problem of "The Quadrature of the Circle."

In the first instance, you were of opinion, that the fallacy which pervaded my argument in support of the

theory, that the ratio of diameter to circumference in a circle, is, as 1 to 3·125, was that of treating linear and square units as if they were identical. I combatted this opinion by furnishing you with a demonstration, in which the comparison of a circle with other geometrical figures, was purely one of line with line, and I now understand you to have abandoned your attack from this particular point, as there is not the slightest reference to it in your letter of the 3rd inst.

In that letter you take up a new position, and charge me with arbitrarily assuming that I know the ratio of diameter to circumference in a circle, and by so doing, take for granted that which has to be proved.

It appears to me, that your admissions have somewhat narrowed the enquiry into the question at issue between us, as they furnish a starting point, upon which our opinions are in unison.

We are now agreed in opinion to the following extent: If it can be shewn that such a circle can exist, as one of which the ratio of diameter to circumference is, as 1 to 3·125, a rectilinear figure equal to it can be described, and that if there be such a circle, the whole problem is solved.

I will make another effort, by a somewhat different train of reasoning from that adduced either in my pamphlet, or hitherto in our correspondence, to prove that such a circle can and does exist, and I am not without hope I shall be able to satisfy *you* in this respect, and obtain from *you* the admission, that the problem of "The Quadrature of the Circle" has been satisfactorily solved.

It is not necessary, I should say, whether I have or have not read at any time, an elementary treatise on Trigonometry, but I may give you the following piece of

information:—When a boy, I was taught the Mensuration of Circles by Walkinghame's Tutor's Assistant, and the following things have ever since remained firmly impressed on my memory:—

First. The ratio of diameter to circumference in a circle, as worked out by Van Culen, viz. 1 to 3·141592 &c. (3·1416 being in most cases considered sufficiently accurate).

Second. The diameter of a circle being given, to find the circumference. Rule.—Multiply the diameter by 3·1416, and the product is the circumference.

Third. The circumference and diameter of a circle being given, to find the area. Rule.—Multiply half the circumference by half the diameter, and the product is the area.

Now, you may assume, if you please, that my knowledge of Mathematics is so limited, that I know no more than is implied in the above piece of information. I will endeavour to shew you, that even this limited amount of knowledge "*is*," with a moderate exercise of the reflective and reasoning faculties, sufficient to demonstrate that 1 to 3·125 "*may*" be adopted as the ratio of diameter to circumference in a circle, not as an arbitrary assumption, but on grounds established by facts, which cannot be subverted by the most acute and subtle reasoning.

Let the diameter of a circle be 1. On the data adopted by the "highest authorities," and approved by yourself, the circumference of the circle is 3·1415926 &c., and its area ·78539815 &c. For practical purposes 3·1416 and ·7854, you will admit may be taken as sufficiently accurate.

One of my methods of proceeding with the inquiry

into this interesting question, was to take the above data for a starting point, and with reference to it I observed the following facts.

First. The square of the diameter of the circle, is to its area, as the four sides of a square described on its diameter, to the circumference of the circle. For, as $1^2 : \cdot785398 15 :: 4 : 3\cdot1415926$. Or, as $1^2 : \cdot7854 :: 4 : 3\cdot1416$.

Second. The converse of this proposition. The area of the circle, is to the square of its diameter, as the circumference of the circle, to the four sides of a square described on its diameter. For, as $\cdot785398 15 : 1^2 :: 3\cdot1415926 : 4$. Or, as $\cdot7854 : 1^2 :: 3\cdot1416 : 4$.

Third. Whether the limited or more extended approximative values of the circumference and area of the circle be adopted, if the arithmetical value of the circumference be divided by the arithmetical value of the area, the quotient is 4 exactly. This you may perhaps call a comparison of linear with square units; be it so, it does not and cannot affect the fact, that the arithmetical symbols which represent the circumference, are equal in numerical value, to four times the arithmetical symbols which represent the area of the circle.

These three peculiarities are inherent in the very nature of the relations of a circle of which the diameter is 1. (The two former are true of all circles.) For, the diameter of the circle being 1, we may take any hypothetical data to represent the circumference, without disturbing those relations of the circle, which are involved in these facts, and which cannot by possibility be affected.

For example, By hypothesis let the circumference of the circle be $3\cdot14$.

Then, $\frac{1}{2}(3\cdot14) \times \frac{1}{2}(1), = 1\cdot57 \times \cdot5, = \cdot785$, will be the

area of the circle. In this example we have altered the arithmetical values, which represent the circumference and area of the circle, but have not disturbed those relations of the circle, which are involved in the facts referred to.

At this point of the enquiry I had to deliberate for some time, but after considerable reflection it occurred to me, the diameter of the circle being 1, and on any data the circumference being exactly equal in numerical value to four times the area, that if the data were correct, the diameter divided by the numerical values of the circumference and area, should produce quotients in the relation to each other of 1 to 4. On trying this, I found it was very nearly, but not exactly so. For, 1 ÷ 3·1416 = ·318309 &c., 1 ÷ ·7854 = 1·27323 &c., and 1·27323 ÷ ·318309 = 3·99998 &c., or very nearly 4, and the relation between the two quotients, is a close approximation to the ratio of 1 to 4.

By repeating this operation a few times, gradually reducing the value of the symbols representing the circumference of the circle, I found at every step, I obtained a nearer approximation to the exact ratio of 1 to 4. It became evident to me that if such relations existed and could be discovered, the value of the circumference would be represented by figures which would divide into 1 without remainder, and by dividing 1 by arithmetical symbols, and gradually reducing the value, I ultimately discovered that the figures 3·125 furnished this desideratum, and that no other figures intermediate between these and 3·1416, would produce the same result. And further, that these figures being taken to represent the circumference of a circle of which the diameter is 1, the diameter divided by the numerical values of the circum-

ference and area, produce quotients which are exactly in relation to each other, in the ratio of 1 to 4, and their values are expressed in figures, which will divide into each other without remainder.

For example: Let 3·125 be taken to represent the circumference of a circle of which the diameter is 1. Then, $\frac{1}{2}$ (3·125) × $\frac{1}{2}$ (1), = 1·5625 × ·5, = ·78125, will be the area of the circle, and the circumference and area are both represented by figures, which will divide into all perfect decimal numbers without remainder. For, 1 ÷ 3·125 = ·32 and 1 ÷ ·78125 = 1·28, and the two quotients are in the ratio of 1 to 4, and will divide into each other without remainder. For, 1·28 ÷ ·32 = 4 exactly, and ·32 ÷ 1·28 = ·25 exactly. And it is true of all circles on this hypothesis, that if the square of the diameter be divided by 3·125 and ·78125, the two quotients are in the ratio of 1 to 4, and will divide into each other without remainder.

For example: Let the diameter of a circle be 9. Then, 9^2 = 81 will be the square of the diameter, 81 ÷ 3·125 = 25·92, and 81 ÷ ·78125 = 103·68; and 103·68 ÷ 25·92 = 4 exactly, and 25·92 ÷ 103·68 = ·25 exactly, and the quotients are the same as in the previous example. The two quotients so obtained, are, under all circumstances, unchangeable quantities, and the product of these two quotients is unity.

If figures of less value than 3·125 be taken to represent the circumference of a circle, these peculiar relations are destroyed, but the variation is in an opposite direction.

For example: Let 3·1248 be supposed to represent the circumference of a circle of which the diameter is 1.

Then, $\frac{1}{2}$ (3·1248) × $\frac{1}{2}$ (1), = 1·5624 × ·5, = ·7812, will be the area of the circle; 1 ÷ 3·1248 = ·3200204 &c., and

$1 + \cdot7812 = 1\cdot2800819$ &c., and $1\cdot2800819 + \cdot3200204 = 4\cdot0000009$ &c., and is in excess of 4, but only to an extent represented by the seventh place of decimals.

I now beg to direct your attention to the following facts, my next discovery in the course of the enquiry :—

The figures $3\cdot125$ being supposed to represent the circumference of a circle of which the diameter is 1, if the square of the diameter of the circle be divided by the circumference, and the circumference be divided by the square of the radius, the product of the two quotients is exactly 4; and this is true of all circles on this hypothesis, with this exception, that the circumference of the circle being divided by the square of the radius, the quotient will not, under all circumstances, be a finite quantity, in which case, an approximation only to 4 is obtained, but this approximation may be carried as close as we please.

For example: Let the diameter of a circle be 1. Then, $1^2 \div 3\cdot125 = \cdot32$, and $3\cdot125 \div \cdot5^2, = 3\cdot125 \div \cdot25, = 12\cdot5$, and $\cdot32 \times 12\cdot5 = 4$ exactly.

Let the diameter of a circle be 9. Then, $9 \times 3\cdot125 = 28\cdot125$ will be the circumference of the circle, and $9^2 = 81$ will be the square of the diameter, and $81 \div 28\cdot125 = 2\cdot88$, and $28\cdot125 \div 4\cdot5^2, = 28\cdot125 \div 20\cdot25, = 1\cdot3888888$ a repeating decimal, and $1\cdot3888888 \times 2\cdot88 = 3\cdot9999999744$, or 4 very nearly, and the decimals may be extended without terminating, *ad infinitum.*

Notwithstanding the length to which this letter has extended, I must direct your attention to one illustration more, my last discovery in this particular branch of the enquiry :

Let the diameter of a circle be 4. Then, $4 \times 3\cdot125 = 12\cdot5$, will be the circumference of the circle ; $\frac{1}{2}(12\cdot5) \times \frac{1}{2}(4), = 6\cdot25 \times 2, = 12\cdot5$, will be the area of the circle,

and the values of the circumference and area, are represented by the same arithmetical symbols.

Let the diameter of a circle be $1{\cdot}28$. Then, $1{\cdot}28 \times 3{\cdot}125 = 4$, will be the circumference of the circle. $\frac{1}{2}(4) \times \frac{1}{2}(1{\cdot}28), = 2 \times {\cdot}64, = 1{\cdot}28$, will be the area of the circle, and the values of the diameter and area are represented by the same arithmetical symbols.

Then, $12{\cdot}5 \div 1{\cdot}28 = 9{\cdot}765625$, and $\sqrt{9{\cdot}765625} = 3{\cdot}125$, the figures which I believe to be the true circumference of a circle of which the diameter is unity.

In this communication I have studiously confined the illustrations of my theory, to those relations of a circle which are inherent in its very nature, all the facts and demonstrations brought under your notice, being entirely independent of the comparison of a circle with any other geometrical figure, and I venture to anticipate you will now abandon your new position of attack, and make the admission that I have not arbitrarily assumed the ratio of 1 to 3·125, in the inquiry,—"Are there any commensurable relations between a circle and other geometrical figures?" But have established, by incontrovertible facts, good and substantial reasons for adopting these figures as an hypothesis, in the consideration of the question.

What followed! I almost immediately discovered commensurable relations between a circle and other geometrical figures, of a most interesting character. One of these you admit, in which the perimeter of a geometrical figure, is to the circumference of a circle, in the ratio of 24 to 25.

There are many others; permit me to direct your attention to the following:—

Describe squares on the sides of an equilateral triangle,

inscribed in a circle. The area of the squares, is to the area of the circle, in the ratio of 24 to 25.

Inscribe a hexagon in a circle. The perimeter of the hexagon is to the circumference of the circle, in the ratio of 24 to 25.

Inscribe a dodecagon within a circle. The area of the dodecagon is to the area of the circle, in the ratio of 24 to 25.

Pray examine for yourself, and then tell me whether these things are not so.

In conclusion I would earnestly beg of you to digest carefully the whole of our correspondence, in connection with my pamphlet, and having done so, I would ask you to favour me with a candid answer to the following question:—

Has there not been introduced into your mind, at least a lurking suspicion, that the problem of "The Quadrature of the Circle," has been satisfactorily solved.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

26th August, 1860.

SIR,

In a hurried visit I paid to London yesterday, I found a letter from you, and also a card. I am sorry I was not at home when you called, because ten minutes conversation on such a subject as the rectification of the circle, is worth dozens of letters.

You remark on my abandonment of the first objection I took to your argument, namely, the confusion of linear and square units, but I beg to explain that I have not done so. This objection referred only to your printed pamphlet, in noticing which, I remarked upon it, and to avoid the stumbling block, suggested that we should confine the argument solely to the question of ratio of diameter to circumference, a question which tests the value of your hypothesis quite sufficiently, and without the liability to double entendre, inasmuch as that linear units, and these alone, are the subject of inquiry.

With reference to "Walkinghame's Tutor's Assistant," all the ratios you quote, only give an approximation to the result. What I want you to shew, is, that there can be a circle whose diameter is 1, and whose circumference is, 3·125 accurately. You give numerous instances of what happens *when* there exists a finite assignable ratio between diameter and circumference. Of course, is my reply. Once admit that the circumference and diameter can be measured by the same linear unit, and the whole problem is solved. Unfortunately if the circumference is 3·125 exactly, the diameter is no longer 1, a fact which you may practically verify.

I would willingly enter into the details of what you adduce as proofs, but it is useless to discuss figures which are based upon an originally erroneous assumption, especially as it is so easy by measurement to prove that assumption is wrong.

Inscribe a regular hexagon in a circle. You will find its perimeter three times exactly the diameter of the circle. Go on increasing the number of sides, making a dodecagon, quindecagon, &c. &c. If you take an area large enough to shew small differences appreciably, as

say 100 feet radius, you will find that the inscribed regular polygon of 20 sides, will pretty nearly coincide with your figures of 3·125; at all events, as you increase the number of sides, and approximate nearer and nearer to the circular perimeter, you will find the perimeter of your polygon approaching nearer and nearer to 3·14159 &c. The more sides you take the more decimals you get. By placing stakes on a flat field, and then straining a line from stake to stake, it is easy to obtain a far more convincing result, than any argument will ever produce upon your mind.

In fact I do not see how argument on such a subject, can be conducted, where one of the arguers is obviously unacquainted with trigonometrical reasoning. Every school boy of 16 who has been at a Mathematical Academy, knows how to find "Circular measure," and cannot know this much, without also knowing and *seeing why* it is impossible the circular circumference should be commensurable with its diameter, just as he knows that the diagonal of a square is not and cannot be commensurable with one of its sides. Any boy who were to *commence* his reasoning by saying, "Let there by hypothesis, be a circle with ratio 1 to 3·125," would be stopped the trouble of further reasoning, by being reminded that it is easily demonstrable that no such ratio *can* exist. Perhaps you would think this an arbitrary stretch of the "Authorities," but at least you must admit, that the school boy in question might be called upon fairly and fully to *investigate* the very simple reasons on which a well known proof is founded, before he undertakes to deny it.

I perceive that you are under the impression that I admit that there is a geometrical figure whose perimeter

is to that of a circle as 24 to 25. Of course I do. But I do not admit that the diameter of that circle can be expressed in finite terms. If again, you assign a finite value to the diameter, away goes your circumference into decimals without end.

Neither can I assent to the proposition that a hexagon inscribed in a circle has its perimeter to that of the circle as 24 to 25. For I know the ratio to be inexpressible in finite terms, since hexagonal perimeter is to diameter, as 3 to 1, and so for the other examples.

In conclusion allow me to say, that I was not aware till lately, how very widely your Pamphlet has been disseminated, for several persons have spoken to me on the subject, and not one who understood the matter but what expressed his wonder that any man with so much knowledge of arithmetic and geometry as the pamphlet exhibits, should be so totally unacquainted with the proof of a ratio, which is as obvious and evident to those who will examine it, as that two and two make four.

For myself, I took the highest mathematical degree which can be taken at the University, many years ago, and have since kept up my knowledge by giving a prize annually for which I examine the boys of a large school myself, and I can pledge the whole of the knowledge I possess, that the idea of a finite ratio such as you have assumed, is simply a misconception, and if any "Mathematician" has assured you that it is a discovery, and that the "Authorities" are all in the wrong, depend on it that he is also a wag.

Either go through a course of plain trigonometry, *or*, what would be a far more easy task, take a measuring tape and a dozen stakes, and you will yourself admit that there is a mistake *somewhere*, in your reasoning, because

the conclusion will not tally with your experiment. Examine attentively, and you will find that in every one of your instances you practically assume a ratio which cannot exist, an assumption which vitiates all your subsequent deductions. I shall be glad to hear you have adopted this test.

I am, Sir,

Yours obediently,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 28th Aug., 1860.

SIR,

I have the honour to acknowledge the receipt of your favour of the 26th Inst.

I do not regret not finding you at home the day I called, for I suspect the ten minutes' conversation would have put an end to our, *to me*, instructive, though I fear to you somewhat troublesome correspondence.

Your proposed suggestion of the measuring tape and dozen stakes, is quite familiar to me, but you must be aware that by such an operation, we should describe a figure outside of, and consequently larger than, the circle we are enquiring about, and I dispute the admissibility of such evidence into the enquiry.

I have previously stated, and now venture to repeat, that Trigonometry, to which you have so often referred, cannot be made available to settle the point in controversy between us, in the way you wish it to be applied, for the simple reason, that the correct circular measure of

an angle, is dependant on the true ratio of diameter to circumference in a circle being known, which is the very point to be decided.

I think, however, I may in the following manner, call in the aid of Trigonometry to assist us in the consideration of the question.

I presume you will admit, that the circular measure of an angle is founded on the two well known geometrical propositions—

First, That in circles of the same radius, the angle is proportional to the arc which subtends it.

Second, That for the same angle, in circles of different radii, the arc varies as the radius.

Then, If we take the unit of angle to be the angle which is subtended by an arc of the same length as the radius; in this case, the angle being measured by the ratio of two lines, it can enter a calculation in which we are dealing with lines.

Let X equal the number of degrees required. Then, X is subtended by an arc of the same length as the radius, equal to R. Now, an angle of 180° is subtended by a semi-circle, that is, by an arc equal to P × R, where P is equal to 3·14159 &c. you would say, but which I maintain to be equal to 3·125.

Then, on your data, $\frac{X^\circ}{180^\circ} = \frac{R}{P\,R} = \frac{1}{P}$; Therefore, X $= \frac{180^\circ}{3{\cdot}14159} = 57^\circ{\cdot}29582$ &c. will be the value of the arc, and equal to the radius; $57^\circ{\cdot}29582 \times 2 = 114^\circ{\cdot}59164$, will be the value of the diameter; and $114^\circ{\cdot}59164 \times 3{\cdot}14159 = 359^\circ{\cdot}\,999950376$, will be the value of the circumference; but is not in agreement with the number of degrees into which the circumference of the circle is divided.

On the writer's hypothesis— $\frac{X^\circ}{180^\circ} = \frac{R}{P\,R} = \frac{1}{P}$; Therefore, $X = \frac{180^\circ}{3 \cdot 125} = 57^\circ \cdot 6$ exactly, will be the value of the arc, and equal to the radius; $57^\circ \cdot 6 \times 2 = 115^\circ \cdot 2$, will be the value of the diameter; and $115^\circ \cdot 2 \times 3 \cdot 125 = 360^\circ$ exactly, will be the value of the circumference; and is in perfect agreement with the number of degrees into which the circumference of the circle is divided, and no other value of P intermediate between 3 and 3·14159, (3·125 of course excepted) will produce the same result.

In the one case all is simple, harmonious, and thoroughly self-consistent, the other commences in mystery and ends in confusion, and it appears to me impossible to doubt, that the true value of P is 3·125.

Permit me to direct your attention to the question in another form.

I think you will not dispute the following formula: Let A represent the area of a circle, C the circumference, D the diameter, and P the number of times the diameter is contained in the circumference, so that $C = P \times D$.

Then, in every Circle, $A = \frac{C \times D}{4}$; Hence—

First. If D be 4, $A = C$. Second. If C be 4, $A = D$. You will at once perceive that both these propositions are facts, inherent in the very nature of the relations of a circle, and cannot be affected by a false value of P. For, Let $P = 3 \cdot 1416$, and $D = 1$. Then, $P \times D = C = 3 \cdot 1416$. $\frac{C \times D}{4} = A = \cdot 7854$. Or, Let $P = 3 \cdot 125$, and $D = 1$. Then, $P \times D = C = 3 \cdot 125$. $\frac{C \times D}{4} = A = \cdot 78125$, and in either case the numerical value of C, is equal to four times the numerical value of A, and would be so if any other arithmetical symbols were taken to represent the

value of P. This, therefore, affords no proof, *per se*, that either one or other of these hypothetical data is correct.

I must now direct your attention to some facts which may be new to you.

Let $D = 1$, $\frac{D}{C} = X$, $\frac{D}{A} = Y$. Now, it can require no effort on your part to perceive, that D being equal to 1, and on any hypothetical data, the numerical value of C, being exactly equal to four times the numerical value of A, the following is a necessary consequence:

If the correct value of P be known, $\frac{Y}{X}$ must be exactly equal to 4, or in other words, the numerical value of Y, must be exactly equal to four times the numerical value of X.

Then, Let $P = 3{\cdot}1416$, and $D = 1$. $P \times D = C = 3{\cdot}1416$. $\frac{C \times D}{4} = A = {\cdot}7854$. $\frac{D}{C} = X = {\cdot}31830914$ &c. $\frac{D}{A} = Y = 1{\cdot}27323656$ &c. And $Y \div X = 1{\cdot}27323656$ &c. $\div$ ${\cdot}31830914$ &c. is so nearly equal to 4, that it would appear to be exactly so, if Y and X were not interminable decimal fractions.

Again, let $P = 3{\cdot}125$, and $D = 1$. $P \times D = C = 3{\cdot}125$; $\frac{C \times D}{4} = A = {\cdot}78125$; and on this hypothesis C and A are represented by arithmetical symbols, which will divide into all perfect decimal numbers, and will also divide into each other, without any remainder. For, $\frac{D}{C} = X = {\cdot}32$. $\frac{D}{A} = Y = 1{\cdot}28$. $Y \div X = 1{\cdot}28 \div {\cdot}32 = 4$ exactly. $X \div Y = {\cdot}32 \div 1{\cdot}28. = {\cdot}25$ exactly. $4 \div {\cdot}25 = 16$ exactly. and $4 \times {\cdot}25 =$ unity. And on this hypothesis these facts are true of all circles, and the following is the reason.

The square of the diameter of any circle, is to its area, as the perimeter of a square described on the diameter of the circle, to its circumference. Or, the converse of this proposition. The area of any circle, is to the square of its diameter, as the circumference of the circle, to the perimeter of a square described on its diameter. These facts may be demonstrated on any hypothetical data taken to represent the value of P.

The following illustrations are deserving of your attention in the consideration of this question :—

Let $P = 3{\cdot}1251$, and $D = 1$. Then, $P \times D = C = 3{\cdot}1251$; $\frac{C \times D}{4} = A = {\cdot}781275$. $\frac{D}{C} = X = {\cdot}31998976$ with a remainder of ${\cdot}00001024$. $\frac{D}{A} = Y = 1{\cdot}27995904$ with a remainder of ${\cdot}00001024$. And ${\cdot}31998976 + {\cdot}00001024 = {\cdot}32$. And $1{\cdot}27995904 + ({\cdot}00001024 \times 4) = 1{\cdot}27995904 + {\cdot}00004096 = 1{\cdot}28$.

Again, Let $P = 3{\cdot}1249$, and $D = 1$. Then $P \times D = C = 3{\cdot}1249$; $\frac{C \times D}{4} = A = {\cdot}781225$. $\frac{D}{C} = X = {\cdot}32001024$ with a remainder of ${\cdot}00001024$. $\frac{D}{A} = Y = 1{\cdot}28004096$ with a remainder of ${\cdot}00001024$. And ${\cdot}32001024 — {\cdot}00001024 = {\cdot}32$. And $1{\cdot}28004096 — ({\cdot}00001024 \times 4) = 1{\cdot}28004096. — {\cdot}00004096 = 1{\cdot}28$.

I think you will not be able to subvert these facts, and that being so, I suspect you will pause before you venture to deny the inference to which they lead, viz. That $P = 3{\cdot}125$, and cannot be correctly represented by any other arithmetical symbols.

May I entreat you to consider these facts in connection with some other facts to which I have already

directed your attention in a previous letter, viz. That if D be 4, the values of C and A, are represented by the same arithmetical symbols. And, that if C be 4, the values of D and A, are represented by the same arithmetical symbols. In the former case, on the writer's hypothesis, the symbols are 12·5, and in the latter 1·28; and $12·5 \div 1·28 = 9·765625$; and $\sqrt{9·765625} = 3·125$; and I can find nothing in any of your arguments, at all tending to shake my conviction, that these figures represent the true circumference of a circle of which the diameter is unity.

You assume that I am totally unacquainted with your methods of proof, of the ratio of diameter to circumference in a circle, and which appears to you to be as obvious and evident to those who will examine them, as that two and two make four. I may say, I have in my own way examined the various proofs given on this point, and they are neither obvious nor evident to me, and I think I can demonstrate why they are not so, when the proper time for doing so arrives.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 31st Aug., 1860.

SIR,

My letter of the 28th inst. extended to such a length, that I had to bring it to a close, without remarking on some of the observations in your letter of the 26th inst., which I think require notice.

Before doing so, however, permit me to direct your attention to an omission made by me when speaking of the circular measure of an angle, and which I consider essential to the completion of my argument.

I maintain that if P represents the number of times the diameter of a circle is contained in the circumference, the true circular measure of an angle is dependant on the true value of P being known. Then, let the value of P be 3·14159. The unit of angle is assumed to be the angle which is subtended by an arc of the same length as the radius, and on this data the length of the arc is represented by the arithmetical symbols $57^\circ{\cdot}29582$ &c., which I maintain to be too little, and being taken to represent the length of the radius of the circle, of which the arc forms part of the circumference, this error is compensated by an error in an opposite direction, namely, by a false value of P, which is in excess of its true value, and so, we arrive at a very close approximation to the values of the circle or semi-circle, but can never arrive at the exact value of either.

This is a point of vital importance in the controversy between us, and justifies me in directing your attention to it in every possible form. I shall, therefore, try to further explain it, by some additional illustrations.

Let the value of P be 3·1251. Then, $\frac{X^\circ}{180^\circ} = \frac{R}{P\,R} = \frac{1}{P}$; therefore, $X = \frac{180^\circ}{3{\cdot}1251} = 57^\circ{\cdot}59815$ &c., will be the value of the arc, and equal to the radius; $57^\circ{\cdot}59815 \times 2 = 115^\circ{\cdot}1963$, will be the value of the diameter; and $115^\circ{\cdot}1963 \times 3{\cdot}1251 = 359^\circ{\cdot}99995713$, will be the circumference; and we arrive at a very close approximation to the value of the circumference of the circle, but on this

hypothesis, I maintain that the value of the arc ($57^\circ\cdot59815$ &c.) is shorter than the radius.

Again, Let the value of P be 3·1249. Then, $\frac{X^\circ}{180^\circ} = \frac{R}{PR} = \frac{1}{P}$; therefore, $X = \frac{180^\circ}{3\cdot1249} = 57^\circ\cdot60184$ &c., will be the value of the arc, and equal to the radius; $57^\circ\cdot60184 \times 2 = 115^\circ\cdot20368$, will be the value of the diameter; and $115^\circ\cdot20368 \times 3\cdot1249 = 359^\circ\cdot999979362$, will be the circumference; and we again arrive at a very close approximation to the value of the circumference of the circle, but on this hypothesis, I maintain that the value of the length of the arc ($57^\circ\cdot60184$ &c.) is longer than the radius, and the arithmetical mean between these two hypothetical values of P, namely, 3·125, is in my opinion the true value of P, and by which alone, we can obtain the true length of the arc, and the exact value of the circumference of the circle. I may here remark, that either of these hypothetical data gives a closer approximation to the circumference of the circle, than if 3·14159 be taken to represent the value of P.

It is just possible you might refer me to the fact, that if we take the value of P to be 3, we can arrive at the exact value of the circumference of the circle. For, $\frac{X^\circ}{180^\circ} = \frac{R}{PR} = \frac{1}{P}$; therefore, $X = \frac{180^\circ}{3} = 60^\circ$; and $60^\circ \times 2 = 120^\circ$; and $120^\circ \times 3 = 360^\circ$ exactly, the circumference of the circle.

Supposing you to do so, my reply is, this would be fatal to your own admission, that hexagonal perimeter, is to the diameter of a circumscribed circle, as 3 to 1, besides, on this assumption, there can be no doubt, the measuring tape and dozen stakes, or, a piece of string and a tolerably round carriage wheel, would be sufficient to

demonstrate the fallacy. I may conclude this part of the subject by stating, that you may take any arithmetical symbols you like, intermediate between 3 and 3·14159 (3·125 of course excepted), to represent the value of P, but you cannot obtain a value of the arc, which taken as the radius of the circle, will carry you to the exact circumference of it, a known, admitted, and finite quantity.

The point in dispute between us may be stated in the following terms:—

You say, "That if the circumference of a circle be expressed in finite terms, the diameter cannot be so expressed. And if a finite value be assigned to the diameter, away goes the circumference into decimals without end." I on the contrary maintain, that both circumference and diameter of a circle, can be truly expressed in finite terms.

Now, it appears to me that your assumption involves an absurdity, and I will proceed to offer some reasons for my thinking so.

Let me refer you to the enclosed diagram No. 1, (see Fig V.) This diagram describes the motion of a wheel in its course through one revolution from A to B along the plane A B. At starting, a point on the wheel marked O, will rest on the plane A B at A, and the line O P, will describe the diameter of the wheel. When the wheel has performed half a revolution, the point O will have passed to C, describing in its course the arc A C, and the point P will have passed to D, describing in its course the arc M D, and the wheel will rest on the plane A B at the point D, having rolled along the plane A B, a distance equal to half its circumference. In the entire revolution the point O will have passed from A to B, describing in its course the cycloidal arc A C B, and the length of the

FIGURE V.

(Diagram 1)

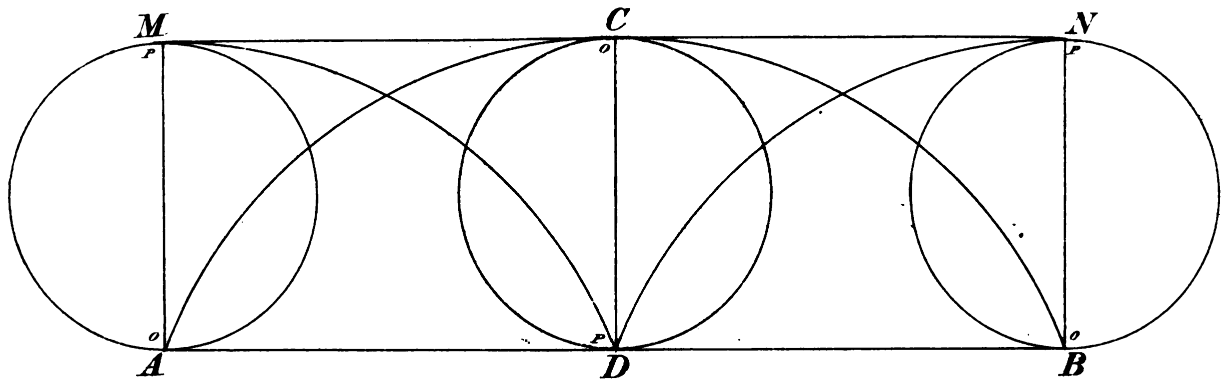
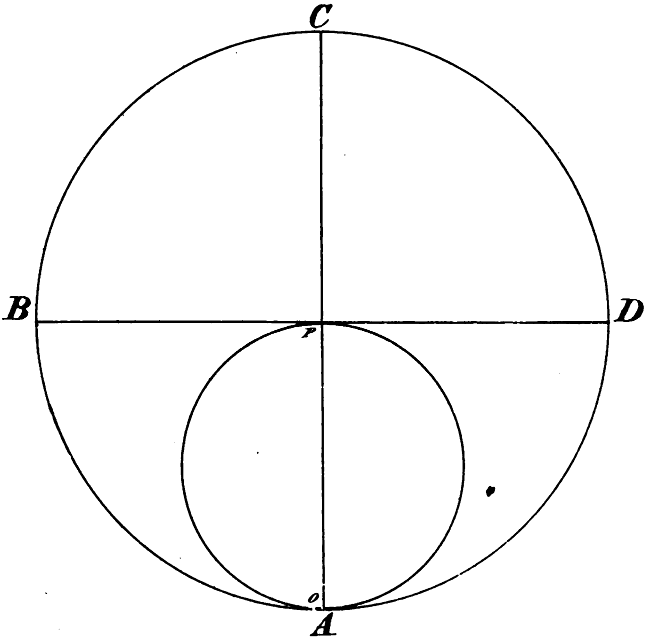


FIGURE VI.

(Diagram II.)



plane A B, along which the wheel has rolled, must be exactly equal to the circumference of the wheel. The arcs A C and C B, are each equal to two diameters of the wheel, and the cycloidal arc A C B is, of course, equal to four diameters. The superficial area of the rectangle A M N B, is exactly equal to 4 times the superficial area of the wheel or circle; and may be demonstrated to be so, either on a true or false value of P (P being taken to represent the number of times the diameter of a circle is contained in the circumference). Then, if the diameter of the wheel be 1, the length of the cycloidal arc will be 4, and there is an exact ratio between the two, of 1 to 4, both finite terms.

These facts will be admitted by every mathematician who has examined the properties of this geometrical figure. The diameter and circumference of the wheel may be called the producing lines, and the cycloidal arc A C B and the plane A B, may be called the produced lines, and it appears to me to involve an absurdity, to suppose there can be an exact ratio between the diameter and one of the produced lines, which can be expressed in finite terms, and no definable relation between the diameter and the other produced line, the two lines being produced, *pari passu*, by the same revolution of the wheel. I venture to put the following question: If there be no definable relation between the diameter and circumference of the wheel at starting, is it conceivable that such a line as the arc A C D, could be described and have a relation to the diameter of the wheel, expressible in finite terms? To me, I confess the thing appears impossible.

Let me now direct your attention to diagram No. 2, (see Fig. VI.) which may be new to you, as I have never met with any mathematician who has thought of con-

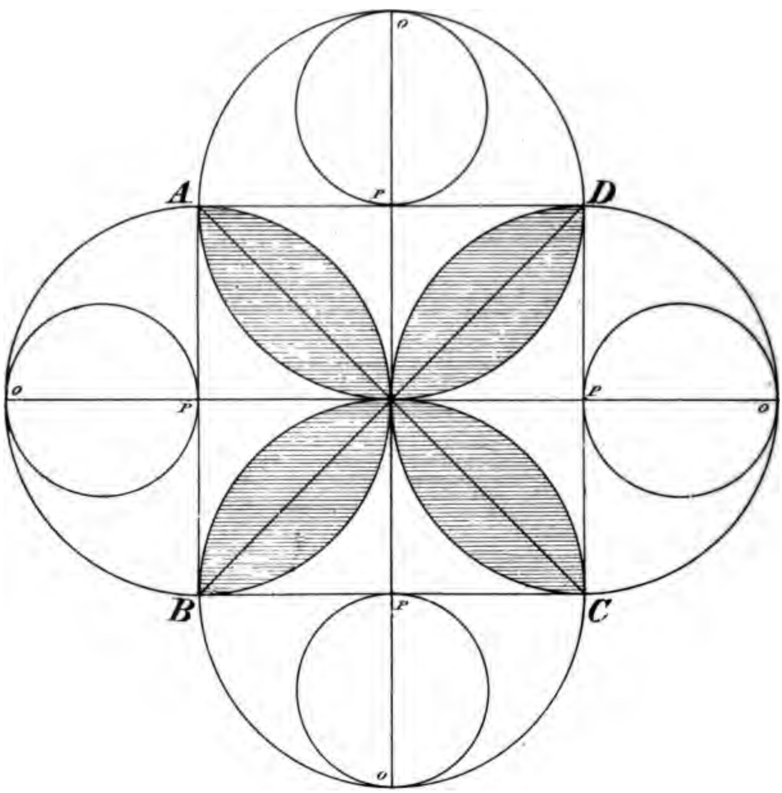
sidering this geometrical figure, with reference to the quadrature and rectification of the circle.

You will perceive that the diameter of the Circle A B C D, is equal to twice the diameter of the inner circle, and consequently, the circumference of the Circle ABCD, must be equal to twice the circumference of the inner circle. Let the inner circle be supposed to revolve round the inner rim of the Circle A B C D, in the direction from A to B C D and on to A again, (it may be supposed to move in either direction). When the inner circle shall have made one revolution, the point O which rested on the point A at starting, will have passed from A to C, describing in its course the straight line A C. The point P of the inner circle which stood, at starting, in the centre of the Circle A B C D, will have passed from the centre to the point B, and back again to the centre. In the next revolution the point O will have passed from C to A, returning on the straight line C A, and the point P will have passed from the centre to the point D, and back again to the centre, and the inner circle will again stand at its original starting point, and in ten thousand revolutions the two points O and P, would go on passing backwards and forwards continually, along the straight lines A C and B D as just described, and could never, by any possibility, get off them.

Then, the described lines A C and B D cut the Circle A B C D into 4 equal parts, each of which is exactly equal in superficial area to the superficial area of the inner circle, and the points O and P, have travelled a distance exactly equal to twice the diameter of the Circle ABCD, and 4 times the diameter of the inner circle, and the ratio of diameter of the inner circle, to the distance travelled by the points O and P, is exactly as 1 to 4.

FIGURE VII.

(Diagram. III.)



I must now direct your attention to diagram No. 3, (see Fig. VII.) This diagram represents four such geometrical figures as that described in diagram No. 2. Let all the four figures be supposed to be set in motion simultaneously by equal forces, so that each would perform the revolutions I have described in explaining diagram No. 2. The points O in the smaller circles, would unite after one revolution in the centre of the square A B C D, the points P having passed to A B C D, and back again to their starting point. At the end of the second revolution, the points P will have passed to B C D A and back again, and the points O will have returned by the way they went, and all will again stand at their original starting point. By their combined action they will have described the square A B C D, the four smaller squares into which the square A B C D is subdivided, and the two curved lines within each of the four smaller squares. The diameters of the larger circles are equal to the side of the square A B C D, and the square A B C D is equal to a square circumscribing the larger circles. The diameters of the smaller circles are equal to the sides of the smaller squares, which squares are equal to a square circumscribing the smaller circles. This is a beautiful demonstration of the well known geometrical facts, that a square described on any line, is equal to four times a square described on half that line; and, a circle described with any diameter, is equal to four times a circle described with half that diameter.

In explaining diagram No 2, I have shewn that the larger circle is cut into four equal parts, each of which is equal in superficial area to the superficial area of the inner circle, and you will observe on reference to diagram No. 3 that such part of the larger circle is now contained in two ways in each of the smaller squares.

Now, if lines be drawn from A to C and from B to D, we obtain the diagonals of all the squares, and the portion of the smaller squares contained between the two curved lines, is divided by the diagonal of the square into two equal parts.

Let the diameter of the smaller circles be 1.

Then, the side of the smaller squares will be 1, and the area of each will be 1, and, on the writer's hypothesis, the area of each of the smaller circles is ·78125. And 1 — ·78125 = ·21875, will be the value of each part of the smaller squares outside the two curved lines, and ·21875 × 2 = ·4375 will be the value of the two parts. And 1 — ·4375 = ·5625, will be the value of that part of the smaller squares contained within the two curved lines, and this is divided by the diagonal of the square into two equal parts; Therefore, ·5625 ÷ 2 = ·28125 is the value of each of these parts. And ·28125 — ·21875 = ·0625. And ·0625 × 16 = unity. I would entreat you to compare this carefully with what I have demonstrated in pages 10 and 11 of my pamphlet, and in my letter of the 28th Aug.

Will you venture to call the following merely extraordinary coincidences?

The value of the superficial area of that part of the smaller squares contained within the two curved lines, I have shewn to be, on the writer's hypothesis, ·5625. To this add the area of the square. Then, ·5625 + 1 = 1·5625, and this is the superficial area of a circle of which the diameter is $\sqrt{2}$. Again, ·5625 and 1·5625 are both arithmetical symbols, of which you can extract the root without any remainder. For, $\sqrt{\cdot 5625} = \cdot 75$; and $\sqrt{1 \cdot 5625} = 1 \cdot 25$. And the arithmetical mean between ·75 and 1·25 is unity, the value of the diameters of the smaller circles, and the sides of the smaller squares. And by this mode

of calculation the same result is obtained, however peculiar may be the decimal selected, as the value of the diameter of the smaller circles.

The multitude of facts which I have now brought under your notice, appear to my own mind to demonstrate beyond all doubt, that the ratio of diameter to circumference in a circle is, as 1 to 3·125, and I anticipate your own faith in the orthodox data will be shaken. If, however, I shall have failed to convince you of the truth of my theory, I have still the resource left me, of pointing out the fallacy, in the proof which you have hitherto considered so simple, and so well founded, as to be obvious and evident to any one who will examine it.

You refer to my pamphlet having been widely disseminated. In explanation I may say, the subject has for some time past occupied my attention, and I had thought at one time to sit down and write a volume, but it occurred to me about a fortnight before the last meeting of "The British Association for the Advancement of Science," to throw off a pamphlet, merely taking up a few leading points, as it was impossible to go into all the ramifications of the subject within the limits of a pamphlet; and by distributing it among the Members of the Association, I thought I might induce some Mathematician to examine the subject from a new point of view. It also occurred to me to send a copy to each Member of the two Houses of Parliament, and in this way I have distributed about 1500 copies. To you the honour is due, of being the first to address me on the subject, and (with one exception to which I referred in a previous letter, and of which you take notice in your last letter), I have not received a communication from any one till within the last few days.

I can assure you I feel grateful for the part you have

taken ; the correspondence has taught me, what I consider additional demonstrations of the truth of my theory, and as the discovery and advancement of truth is my only object, whatever may be the result, I shall never regret the correspondence, and shall ever bear you in grateful remembrance for inducing it.

I am, Sir,

Very faithfully yours,

JAMES SMITH.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

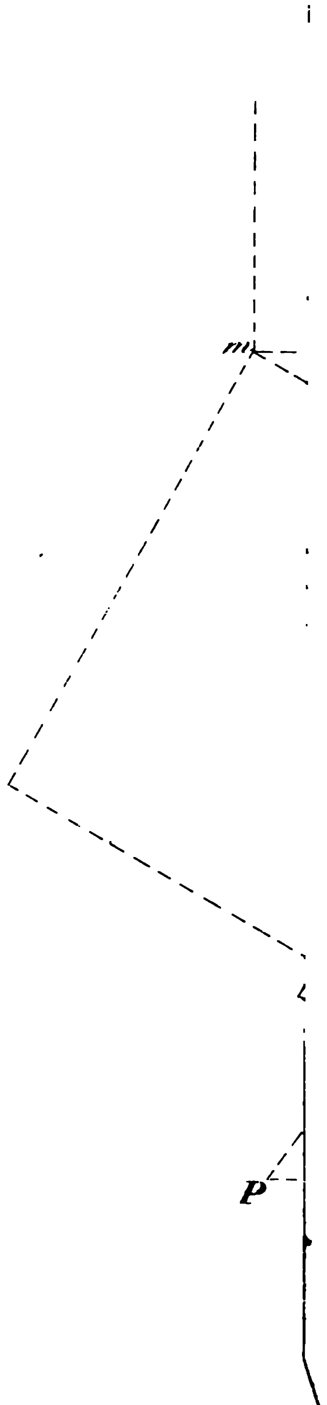
Barkeley House, Seaforth,
Liverpool, 7th Sept., 1860.

SIR,

I wrote you on the 28th and 31st ult., but have not since heard from you.

I presume your engagements at the present time prevent you giving my letters immediate attention; in the mean time, I venture to furnish you with some additional information, which, when you have leisure to resume the consideration of the subject, you may find of some assistance to you. I hope you will be at the trouble of perusing carefully the whole of our correspondence, in connection with my pamphlet, and this I feel assured you will do, as I cannot doubt your feeling, like my own, is one of earnest desire to arrive at the truth, on an important mathematical question, in the settlement of which, the interests of Science are, nay, the interests of mankind may be said in a certain sense to be, concerned.

About six weeks ago I was introduced to a gentleman


m
P

who, like yourself, obtained the highest honours at the University. He had been present at the meeting of the British Association at Oxford, and was one of those to whom I had sent a copy of my pamphlet. We had only a very short conversation, as he thought it was not worth while entering upon a discussion of the question, as there would not be the slightest chance of our agreeing in opinion. I discovered he had not been at the trouble of reading my pamphlet.

I was surprised the other day at receiving a long letter from him, in which he goes at length into the usual proof, for discovering the value of the semicircle of which the radius is 1, by means of the semi-perimeter of a polygon of an infinite number of sides, and assumes that I cannot have had my attention directed to a proof, in his opinion, so clear and obvious, and on my examining of which he feels assured I shall at once abandon, what he calls my erroneous opinions.

In reply I referred him to the enclosed diagram, No. 1, (see Fig. VIII.) for the purpose of pointing out a fallacy, exactly similar to one in his mode of calculation, and the following is the substance of my letter to him :—

"It appears to me there is a fallacy in your mode of ascertaining the ratio of diameter to circumference in a circle, not at first sight apparent, but a fallacy notwithstanding.

Let A G, the diameter of the circle, be 8.

Then, A B C is an equilateral triangle inscribed in the circle, of which the value of the sides is $\sqrt{48}$, and consequently the superficial area of squares described on the sides of the triangle A B C must be 48.

D E F is an equilateral triangle, of which the value of the sides is equal to the radius of the circle. The super-

ficial area of the triangle, is equal to the sixth part of the superficial area of a regular hexagon inscribed in the circle, but the value of which cannot be expressed in finite terms.

D E G F D is a quadrilateral, of which the superficial area is equal to the sixth part of the superficial area of a regular dodecagon inscribed in the circle, the area of which (dodecagon) is exactly equal to the area of a square described on one of the sides of the equilateral triangle A B C. The value of the superficial area of the quadrilateral is therefore 8, and may be demonstrated to be so as follows :—

The triangles D H E, D H F, G H E, G H F, are together equal in superficial area to the rectangle L M G D, of which one side is D G the radius of the circle, and the other side equal to H E, which is equal to half the radius of the circle. Therefore, $\frac{1}{2}$ (8) × $\frac{1}{2}$ (4) = 4 × 2 = 8, is the superficial area of the rectangle L M G D, and is equal to the area of the quadrilateral D E G F D. Then, the value of the line D H, which is common to the two right-angled triangles D H E, D H F, is equal to $\sqrt{ED^2 - EH^2}$, $= \sqrt{4^2 - 2^2}, = \sqrt{16 - 4}, = \sqrt{12}, = 3{\cdot}4342$ &c. The value of the line H G, which is common to the two right-angled triangles G H E, G H F, is equal to D G — D H, = 4 — 3·4342, = ·5658. Then, D H = 3·4342 ; H G = ·5658; H E = 2. And D H × H E, = 3·4342 × 2, = 6·8684; and H G × H E, = ·5658 × 2, = 1·1316; and 6·8684 + 1·1316 = 8, the area of the quadrilateral D E G F D exactly. But observe, (and here lies the fallacy) 3·4342, &c., is an interminable decimal fraction, and is less than the true value of D H ; and ·5658 is greater than the true value of H G; and by your mode of calculation this false value of H G, would be taken as one of the elements to ascertain the value of E G, and you thus get a false

value of E G, and at every successive step this becomes an increasing error, and it is impossible to arrive at the true ratio of diameter to circumference in a circle, by any such method, or even to the close approximation to it which you imagine.

Permit me to direct your attention to the following facts, in connection with this diagram:—

If the circle be circumscribed by a square, and another square be inscribed in it, the area of the former will be 64; and the area of the latter 32. The area of a square described on a side of the equilateral triangle A B C will be 48, and the area of a square described on a side of the equilateral triangle D E F will be 16.

Then, $64 + 32 = 96$, and $96 \div 2 = 48$, and 48 is the arithmetical mean between the two numbers 64 and 32; and is, therefore, the arithmetical mean between the value of the area of a square circumscribed about the circle, and the value of the area of a square inscribed in it, and is equal to the superficial area of a square described on the sides of the equilateral triangle A B C.

Again, $48 + 16 = 64$, and $64 \div 2 = 32$, and 32 is the arithmetical mean between the two numbers 48 and 16; and is, therefore, the arithmetical mean between the value of the area of a square described on the sides of the triangle A B C, and the value of the area of a square described on the sides of the triangle D E F, and is equal to the superficial area of a square inscribed in the circle.

Again, the side B C of the equilateral triangle A B C, cuts A G, the diameter of the circle, at K. The line A K is to the line K G, in the ratio of 3 to 1.

Again, the area of the equilateral triangle A B C, is to the area of the equilateral triangle D E F, in the ratio of 3 to 1.

Again, the area of a square described on a side of the equilateral triangle A B C, is to the area of a square described on a side of the equilateral triangle D E F, in the ratio of 3 to 1.

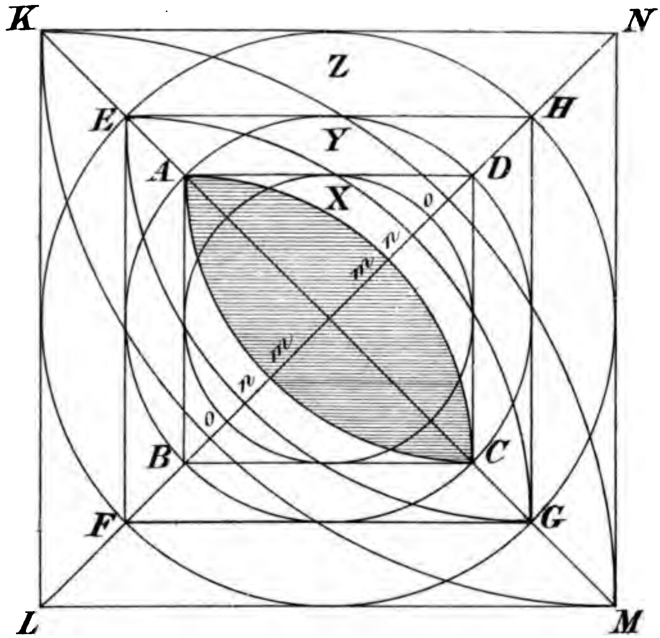
Again, if a line be drawn from an angle, say B, of the equilateral triangle A B C, to a point *p*, in the circumference of the circle, which shall bisect the angle B; and another line be drawn from *p*, the point of contact with the circumference, to C, the line *p* C will be equal to the side of a regular hexagon inscribed in the circle, and is equal to the radius of the circle. The value of the perimeter of the regular hexagon will be 24, and the ratio between the perimeter of the hexagon and diameter of the circle, is as 3 to 1.

Again, if squares be described on the sides of the equilateral triangle A B C, and a line be drawn from an outer angle of one square to the outer angle of another square adjoining it, we obtain the straight line *m n*, in the diagram, and if a square be described on this line, the area of such square is to the area of the squares described on the sides of the equilateral triangle A B C, in the ratio of 3 to 1; and if a circle be inscribed in this square, the operation described by means of this diagram may be repeated in an outward direction. The diameter of the new circle will be equal to the diameter of the generating circle, plus the radius, $= 8 + 4 = 12$. This applies also to the equilateral triangle D E F, in which case the line *o p* is obtained, which is equal to the sides of the equilateral triangle A B C, and the operation may be repeated in an inward direction. The diameter of the new circle, will in this case be $\sqrt{48}$.

Again, a square described on the line D G, is equal to a square described on a side of the equilateral triangle

FIGURE IX.

(Diagram. II.)



D E F, and we can now obtain an equilateral triangle D N E, equal to the equilateral triangle D E F, and which is common to the square described on the line D G, and the square described on the line D E.

Again, we can also obtain a quadrilateral D N B E D, equal to the quadrilateral D E G F D, which is common to the square described on the line D G, and the square described on the line D E, and the quadrilateral D N B E D, is exactly equal in superficial area to half the area of the squares.

You will perceive there is a perfect relation between all these geometrical figures, which can be expressed in finite terms, and if there were no definable relation between the diameter and circumference of a circle, from which, and by means of which, they are all produced, I maintain that this would be an impossibility."

So much for the substance of my letter to the gentleman alluded to. Permit me now to direct your attention to the enclosed diagram, No. 2, (see Fig. IX.) and in connection with it, may I ask you to bear in mind the diagram No. 3, in my letter of the 31st ult., (see Fig. VII.) and my explanation of it. In the square A B C D in this diagram, you will perceive there is an inscribed trapezium, similar to the trapezium inscribed in the smaller squares in the diagram referred to. With reference to this diagram, every mathematician will admit the following facts, viz.: The area of circle Y, is equal to twice the area of circle X, and the area of circle Z, is equal to twice the area of circle Y, and four times the area of circle X. The area of square E F G H, is equal to twice the area of square A B C D, and the area of square K L M N, is equal to twice the area of square E F G H, and four times the area of square A B C D. The diagonal A C or D B of the

square A B C D, is equal to the diameter of circle Y, and the diagonal E G or H F of the square E F G H, is equal to the diameter of circle Z.

Let the diameter of circle X be 1, the side of the square A B C D will be 1, and the superficial area of the square A B C D will be 1, and on the writer's hypothesis, the superficial area of the circle X is ·78125.

Then, the area of square A B C D, minus the area of circle X, $= 1 - {·78125}, = {·21875}$, and this difference is the value of one of the parts of the square A B C D, outside the curved lines A *m* C. There are two such parts within the square. Therefore, $·21875 \times 2, = ·4375$, will be the value of the two parts of the square outside the curved lines A *m* C. Then, the area of the square A B C D, minus the area of the two parts of it outside the curved lines A *m* C, $= 1 - ·4375, = ·5625$, and this must be the area of the trapezium A *m* C *m* A, being that part of the square A B C D, contained within the two curved lines A *m* C. To this add the area of the square A B C D. Then, $·5625 + 1, = 1·5625$, and this will be the superficial area of circle Y, and is equal to twice the area of circle X.

Again, the superficial area of the trapezium A *m* C *m* A is ·5625. The superficial area of the square A B C D is 1. And $·5625 + 1 = 1·5625$, is the superficial area of the circle Y. But, $\sqrt{·5625} = ·75$, and $\sqrt{1·5625} = 1·25$, and $·75 + 1·25 = 2$, and $2 \div 2 =$ unity; and unity is the arithmetical mean between the two numbers ·75 and 1·25, and is equal to the diameter of circle X, and the side of square A B C D.

Again, A B, B C, A D, D C, represent the sides of the square A B C D, and $AB^2 + BC^2$, or $AD^2 + DC^2, = 1^2 + 1^2 = 2$; and $\sqrt{2}$ is the value of A C, the diagonal of the square A B C D; and A C is the diameter of circle

Y, and equal to the side of the square E F G H; and we have now obtained the following values.

The side of square E F G H = $\sqrt{2}$. The diameter of circle Y = $\sqrt{2}$. The area of square E F G H = 2. The area of circle Y = 1·5625. Then, pursuing the same mode of calculation, the area of square E F G H, minus the area of circle Y, = 2 — 1·5625 = ·4375, and ·4375 × 2 = ·875, and this will be the value of the two parts of the square E F G H, outside the curved lines E *n* G. Then, the area of the square E F G H, minus the area of the two parts of the square outside the curved lines E *n* G, = 2 — ·875 = 1·125, and this must be the value of the trapezium E *n* G *n* E, being that part of the square E F G H, contained within the two curved lines E *n* G. To this add the area of the square E F G H. Then, 1·125 + 2 = 3·125, and this will be the superficial area of circle Z; and is equal to twice the area of circle Y, and four times the area of circle X; and the arithmetical mean between the two numbers $\sqrt{1{\cdot}125}$ and $\sqrt{3{\cdot}125}$ is $\sqrt{2}$, and is equal to the diameter of circle Y, and side of square E F G H.

Again, E F, F G, E H, H G, represent the sides of the square E F G H, and E F² + F G², or, E H² + H G², = $\sqrt{2}^2 + \sqrt{2}^2 = 4$; and $\sqrt{4} = 2$, is the value of E G, the diagonal of the square E F G H; and E G is the diameter of circle Z, and equal to the side of the square K L M N; and we have now obtained the following values.

The side of square K L M N = 2. The diameter of circle Z = 2. The area of square K L M N = 4. The area of circle Z = 3·125. Then, the area of square K L M N, minus the area of circle Z, = 4 — 3·125 = ·875, and ·875 × 2 = 1·75, and this will be the value of the two parts of the square K L M N outside the curved lines K *o* M. Then, the area of the square

K L M N, minus the area of the two parts of the square outside the curved lines K *o* M, $= 4 - 1{\cdot}75 = 2{\cdot}25$, and this must be the value of the trapezium K *o* M *o* K, being that part of the square K L M N, contained within the two curved lines K *o* M. To this add the area of the square K L M N. Then, $2{\cdot}25 + 4 = 6{\cdot}25$, and this would be the superficial area of a circle circumscribed about the square K L M N; and equal to twice the area of circle Z, four times the area of circle Y, and eight times the area of circle X. Then, $\sqrt{2{\cdot}25} = 1{\cdot}5$, and $\sqrt{6{\cdot}25} = 2{\cdot}5$, and the arithmetical mean between $1{\cdot}5$, and $2{\cdot}5$, is 2, which is the value of the diameter of circle Z, and side of the square K L M N; and by continuing to circumscribe the outer figure of the diagram, this mode of calculation might be pursued *ad infinitum*.

In this way all is simple and harmonious. We obtain arithmetical values for every line in the diagram, thoroughly consistent with what every Mathematician must admit to be geometrically true, and this would be an utter impossibility, if there were no definable relation between the diameter and circumference of a circle.

Now, take p to represent the number of times the diameter of a circle is contained in the circumference, and let me entreat you to examine what sort of result you would obtain, either on your own or any other value of p; and assuming you to have done so, I will venture to put the following question, and respectfully request you to favour me with a candid answer:

Is it possible to doubt that the true value of p is $3{\cdot}125$?

I am, Sir,

Yours very respectfully,

JAMES SMITH.

P.S.—As I had directed your attention, in a previous letter, to certain relations between a circle and two right-angled triangles, it did not appear to me necessary to introduce the triangles A G P and A R G, into the diagram No. 1, but it occurred to me after I had written this letter, that the relations of a circle to other geometrical figures, as explained by this diagram, were not complete without them; I have, therefore, described them at the last moment, and must refer you for what I have said with regard to them, to my letter of the 30th July.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

Milford Haven, 6th Sept., 1860.

SIR,

I find by your letters received on my arrival here, that you do not approve of my suggestion to measure the semi-circumference of a circle whose radius is unity, in order to obtain a practical proof that (whatever the ratio of circumference to diameter may be) it is certainly not $3\frac{1}{8}$ to 1. This is unfortunate, because if you would make the experiment, it would at once prove, to ocular demonstration, that there is a circle whose diameter does not bear to its circumference the ratio which "the writer's hypothesis," so often pressed into the service of his argument, assumes, and by consequence inevitable, that all circles are subject to the same remark, and therefore no circle can exist such as you have supposed. You may assume the diameter to equal 1, or, you may assume the circumference to equal $3\frac{1}{8}$, but you must not assume *both* values simultaneously. Now, there are the following

remarkable words in your letter, "You are aware that by such an operation (that of actual description of a circle by measuring tape,) we should describe a circle outside of, and consequently larger than, the circle we are enquiring about." My proposal was to measure a radius of 100 feet, and to describe a semi-circumference, in order by measurement of the latter to prove ocularly, that its length is greater than 312·5 feet. If then the semi-circumference of this circle of 100 feet radius, is "outside of and consequently larger than" some other circle, this *other* circle cannot have 100 feet for its radius. In using the expression you have done—"the circle we are enquiring about" —the pervading error is concentrated in a few words. The property of one circle is the property of all circles, and if there is *any* circle whose radius is 1, and whose semi-circumference is greater than 3·125, or 100, and greater than 312·5, then in *all* circles the ratio above-named is greater than the ratio of 1 to 3·125.

You will excuse me if I decline following all the calculations you have made, and all the illustrations you have given, for in every individual example, you have either tacitly by implication, or openly in words, assumed that a circle can be described "on the writer's hypothesis."

If such a circle can be described, *cadit quæstio*, if not, all your examples fall still-born.

Referring to the case of the cycloid, I cannot see your difficulty. The base of a cycloid is equal to the circumference of the generating circle, and therefore if your circle's diameter is unity, your base is not measurable by the assumed unit, in fact it is 3·14159 &c. The area of the rectangle A M N B, is a finite quantity, it is true, but it is not commensurable with the area of the generating

circle, except by the introduction of the old difficulty, viz., an interminable decimal, which the base of our rectangle represents when the diameter is 1.

In the diagrams 2 and 3, I do not see what you prove, except what Euclid has done, that circles are to each other as the squares of their diameters. When you apply this well known property of circles, you fall into the old trap, and go on to say, "The area of each small square is 1, and on the writer's hypothesis, the area of the smaller circle is ·78125." All that follows, depends for its truth on the truth of the "writer's hypothesis." The writer must be good enough to give a demonstration of the truth of his hypothesis, before he expects me to admit conclusions based upon it.

As to pointing out the "fallacy" of the proofs I alluded to as conclusively shewing that the ratio of circumference to diameter, cannot be shewn except in interminable decimals, I would strongly advise you not to waste your time in so hopeless an attempt. Can you seriously suppose that Newton, La Place, Des Cartes, &c., &c., and all the wise and learned philosophers who ever lived, were so grossly ignorant as to deceive themselves, on a matter which is susceptible of the most distinct proof, and may be verified by actual ocular inspection by the humblest individual.

I am, Sir,

Very truly yours,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 13th September, 1860.

SIR,

My last letter was written before yours of the 6th inst. came to hand, the receipt of which I have now the honour to acknowledge.

It did not occur to me while writing my letter of the 7th inst., that it was necessary I should describe in the diagram No. 1, (see Fig. VIII.) the two right-angled triangles A G P and A R G, of which A G, the diameter of the circle is common to both, as I had in a previous letter directed your attention to these figures, and their relations with the circumference of a circle, but perceiving it was necessary to the completion of my argument to do so, I described them on the diagram at the last moment, and in the hurried postscript referring to this, I omitted to direct your attention to the facts, that a square described on a side of the equilateral triangle A B C, is equal in superficial area to twice the area of the triangle A G P, of which A G the diameter of the circle, is the perpendicular; and three and one eighth times the area of the triangle A R G, inscribed in the circle, and of which A G the diameter of the circle, forms the hypotheneuse.

Judging from your last communication, it would seem that it is as little likely you and I can ever arrive by correspondence, at an agreement of opinion on the controverted question of "The Quadrature of the Circle," as that the sine and tangent of an angle should ever become the same straight line; for you candidly admit, that you

"decline following me in all the calculations I have made, and all the illustrations I have given."

I may here mention that I have at different times, adopted several mechanical methods for the purpose of demonstrating the truth on this question, among others that of weight. A circle of a given diameter contains a certain superficial area, and is equal in superficial area to a rectangle of particular dimensions. I have made the comparison between the two in fine rolled metal, with the greatest care, and I may frankly tell you that on the orthodox data, the rectangle supposed to contain the same superficial area as the circle, outweighs the circle; but I have hitherto been twitted by the Mathematician, on the absurdity of all such unmathematical methods, of dealing with a purely mathematical question; yet, recent experience has taught me that the Mathematician himself can now suggest a mechanical test. If I should fail to satisfy you by other means, I may yet take you at your word, and make you a proposal based on your own suggestion.

Before doing so, however, I shall make a few further remarks. Let it be admitted that with any straight line as diameter, a circle may be described. Then, permit me to entreat you to abandon your resolution of not following me in all my illustrations, and if possible to put aside your prejudice for a few minutes, and take in your hand the diagram No. 1, enclosed in my letter of the 7th instant, (see Fig. VIII.) with reference to which, I have to make the following observations.

The circle has a diameter A G, and this line may be of any length.

Then, you must necessarily admit the following facts: That the circle may be circumscribed by a square, and that another square may be inscribed within the circle.

That an equilateral triangle may be inscribed within the circle, and that squares may be described on the sides of the triangle. That a circle may be described within the generating circle, with the line D G, a radius of the generating circle as diameter, and that within this circle an equilateral triangle may be inscribed, and on the sides of it, squares may be described. That two right-angled triangles may be described, of which the diameter of the circle shall be common to both, being the perpendicular of one, and the hypotheneuse of the other. That a regular hexagon and a regular dodecagon may be inscribed in the circle. That an equilateral triangle, and quadrilateral, may be described within the circle, both of which shall be common to two distinct squares, described on radii of the circle. That lines may be thus obtained by which the operation may be repeated inwards or outwards, *ad infinitum*. That supposing the operation to be repeated, every figure and every line of each figure in the new diagrams, will be proportionals of the corresponding figures and lines, connected with the generating circle. That a circle being described, every other figure here referred to may be described, without any reference to the question, What is the relation between the diameter and circumference of a circle ? And, That there is a perfect relation between all these geometrical figures, which can be expressed in finite terms.

Let me now request you to pause, and reflect, on the following questions.

Are not all these geometrical figures emanations from the circle ? Has not each one its boundaries and limits strictly defined by the circle ? Suppose you to take your rule and compass, and make an attempt to describe the figures independently of the circle, do you not at once

feel convinced, that you would certainly fail to accomplish the task?

Then, may I not ask: Is it conceivable that any one, who has permitted his reasoning and reflective faculties to have fair play, would in the face of these, and the multitude of other facts, introduced to your notice throughout our correspondence, and in my pamphlet, attempt to maintain a theory so utterly absurd, as, that there is no definable relation between the diameter and circumference of a circle?

You may not yet be convinced, but I do not hesitate to tell you, that if the definable relation between the diameter and circumference of a circle, "on the writer's hypothesis," be in perfect harmony with every one of these facts, and I defy you to prove to the contrary, then, in spite of all your opposition, such definable relation must necessarily be true.

When it is known that eight circumferences of any circle, are exactly equal to twenty-five diameters, how plain, how simple, how beautiful, everything connected with this interesting question becomes. How consistent and harmonious, not only with common sense, but with the great and glorious works of Him who could say, "Let there be light, and there was light."

I now venture to tell you, that the day is not far distant, when, as certainly as Euclid hypothetically assumed a false line, so certainly will every Teacher of Mathematics assume false figures, such as 3·14159, to furnish his pupils with a "*reductio ad absurdam*" demonstration; on the ratio of diameter to circumference in a circle.

The time has now arrived, when I may point out what appears to me to be the fallacy in the ordinary methods adopted, for ascertaining the relation between the diameter and circumference of a circle.

The following is the usual and most familiar mode.

Let a four-sided polygon be circumscribed about, and another four-sided polygon be inscribed in a circle, of which the radius is 1. The area of the circumscribed polygon will be 4, and the area of the inscribed polygon will be 2.

Now, let it be admitted that such a thing is conceivable as a thin elastic line, of any length, say 32 inches, and let this line be made to describe a rectangle of four equal sides. Each side of the rectangle will then be 8 inches, and it will contain a superficial area of 64 inches.

Can you shew me how this line can be made to describe a rectangle of any other form, and contain the same area? I think not.

If the line be made to describe a rectangle, of which the longer side is 8·00000001, and the shorter side 7·99999999, the four sides of the rectangle will still be together equal to 32 inches; but, $8·00000001 \times 7·99999999 = 63·9999999999999999$; and you may multiply the decimals, "*ad infinitum*," but you could never make the line in this form, to contain an area of 64 inches.

Let the line be made to describe the circumference of a circle. It will enclose a certain area. Can you shew me how it can be made to contain the same area in any other form whatever? I know you can not.

Then, apply these plain, obvious, and I presume admitted principles, to the consideration of the question at issue between us, and mark the results to which you are led, by adopting such methods as those we were taught, of ascertaining the ratio of diameter to circumference in a circle.

The radius of a circle being 1, the area of a four-sided circumscribed polygon will be 4, and the area of a four-sided inscribed polygon will be 2.

Then, $\sqrt{4 \times 2} = \sqrt{8} = 2{\cdot}8284271$ &c., will be the area of an inscribed eight-sided polygon. And, as $(2 + \sqrt{2}) : 4 :: 4 : 3{\cdot}3137085$ &c., and this will be the area of a circumscribed eight-sided polygon; and pursuing this mode of calculation, with which you are of course quite familiar, the (supposed) circumscribed and inscribed polygons, approximate at every successive step more nearly to equality, in the value of their sides and areas. This operation is repeated until it is supposed, a circumscribed and an inscribed polygon are obtained about the circle, of which the areas of both are represented by the figures 3·1415926 &c. These are said to be polygons of 32768 sides.

You were no doubt taught to reason on this as I was, and I presume you still continue to think the reasoning unanswerable. You are certainly so far justified, that you can advance the names of Newton, La Place, Des Cartes, and many other wise philosophers, who thought as you do.

The reasoning I was taught was as follows:—As the area of the circle is intermediate between the areas of the (supposed) inscribed and circumscribed polygons, and as the areas of the polygons are the same, and their values represented by the figures 3·1415926 &c., it follows, that the area of the circle must also be 3·1415926 &c. And the circumference of circles being as their radii, these figures must represent the circumference of a circle of which the diameter is 1. (I may here make a passing remark. One of your charges against me is, that of treating linear and square units as if they were identical. I have not done so, but even if I had, have I not here a precedent for doing it?)

Now, I deny that the area of the circle is intermediate between the areas of the (supposed) inscribed and circum-

scribed polygons; and mark the absurdities to which your conclusion leads. On your principle, we have three geometrical figures: a circle, a figure circumscribing it, and an inscribed figure within the circle ; and yet they are all made to contain the same area. You will admit, on the one hand, that a line in the form of the circumference of a circle, will contain a larger area than the same line can be made to contain in any other form whatever; and yet, on the other hand, you would ask me to believe that a figure inscribed in the circle, the perimeter of which must necessarily be less than the circumference of the circle, can actually contain an area equal to the area of the circle.

It is true, that the original circumscribed and inscribed four-sided polygons, are by this mode of calculation made to meet in a polygon of an infinite number of sides, which is the mean between the two, but this is a polygon of which the perimeter is longer than the circumference of the circle, and contains a larger area than the circle.

Again, if the circle be circumscribed by a polygon, of an infininte number of sides, a side of the polygon will necessarily be the tangent of an angle, and the side of a polygon of an infinite number of sides, inscribed in the circle, will necessarily be the sine of an angle, and this line, if produced, would cut the circumference of the circle. If the areas of the (supposed) circumscribed and inscribed polygons be equal in area to the circle, and the perimeters of the polygons and circumference of the circle also equal :—Pray tell me how can this be possible? Do you wish me to believe that the tangent of an angle (in this case a line outside the circle), and the sine of an angle (in this case a line inside the circle), can become one and the same straight line ?

Again, you say "That the property of one circle is the property of all circles." True. You maintain that the radius of a circle being 1, the area of a circumscribed polygon of 32768 sides, and the area of an inscribed polygon of 32768 sides, are both equal in superficial area, to the area of the circle; and therefore, the perimeter of each of the polygons is equal to the circumference of the circle.

Let the radius of a circle be equal to the earth's distance from the sun, say about 96,000,000 miles—(this I believe to be the exact distance of the earth from the sun)—the circumference of a circle equal to the earth's orbit will be about 600,000,000 miles. And 600,000,000 ÷ 32768, is equal to about 18300 miles, and this would be about the value of the side of a polygon of 32768 sides, of which the perimeter would be equal to the circumference of a circle, of which the radius is about 96,000,000 miles.

Do you ask me to believe that a polygon of 32768 sides, of which the perimeter is 600,000,000 miles, will contain as large an area as a circle, of which the radius is 96,000,000 miles? And yet, if I am to adopt the orthodox data, I presume you would expect me to believe this to be a fact.

It is true, the mean between a four-sided circumscribed polygon, and a four-sided inscribed polygon; about a circle of which the radius is 96,000,000 miles, may be ascertained, but to do so we must not stop at polygons of 32768 sides; to ascertain the mean at all, would be a work costing no trifling amount of time and labour.

Have you ever tried a mechanical experiment on polygons about a circle? If you have not, I have; and I will tell you the result, which, if you will be at the trouble, you may verify for yourself in the following way.

On an even surfaced wood floor, draw a straight line

128 inches long, with this line as diameter describe a circle, and divide the line describing the circle into 100 equal parts; at the points on the circumference of the circle which mark these parts, drive a small tack, and around these tacks place a thread, (a fine wire thread is the best thing for the purpose), you will thus obtain the 100 sides of a polygon inscribed in the circle. Then, with similar small tacks, mark the angle points of a circumscribed polygon, and surround them with a similar thread. You will find the threads describing the sides of the inscribed polygon, and the parts of the circumference of the circle, on which the threads rest, are apparently both straight lines, and you cannot distinguish one from the other, and at this point the area of the circle would appear to be exhausted, and exactly equal to the polygon described by the threads. You will then find that outside the circumference of the circle, and between it and the threads describing the circumscribed polygon, there is a distinct and appreciable quantity, and you must at once perceive, that the mean proportional between the areas of two such polygons, must necessarily give you the value of a figure outside of and larger than the circle.

In conclusion, I may remark, that I would rather find you grappling with my facts and arguments, than sheltering yourself behind the authority of great names; and this I think you are bound to do, as you undertook to prove that my reasoning was based on a fallacy.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, 21st Sept., 1860.

SIR,

On my return here I found your two letters of the 7th and 13th instant.

Courtesy prevents my making the same remarks upon your reasoning, which you have made upon that of every Mathematician in Europe, when you say we " maintain a theory so utterly absurd, as that there is no definable relation between the diameter and circumference of a circle."

Humility, one of the first of qualifications in those who earnestly seek truth, is obviously not one of your characteristics.

All the conclusions you have arrived at, are based on the assumption that "the writer's hypothesis" is correct. No one will dispute, that if the circle you imagine can really be drawn, various definable relations will exist. I have so often told you that no such circle can be drawn, that I feel it is childish to go on repeating the same remark.

It appears now, from your letter, that you are familiar with the popular proof consisting of the polygons inscribed in and circumscribed to a circle, and which by multiplying the number of sides at pleasure, can be made to approximate to (not to equal) the circumference of the circle, and to one another as closely as we please. And that this proof is not satisfactory to your mind.

If it is not clear to you that the inscribed regular polygon is necessarily less than the circle, and the circumscribed always greater than the circle, and that the circle

itself is the limit to which they both *approach*, any further discussion would be a waste of time. Indeed, when you "deny that the area of the circle is intermediate between the areas of the inscribed and circumscribed polygons," in round terms, it is high time to close a discussion of so puerile a nature. Let me then observe, that I have no theory whatever on the subject peculiar to myself, that in my letters I have no object whatever but to point out to you, how you may convince yourself that it is not the world, but yourself, that is mistaken, and that if you cannot see what is obvious to every man who will carefully examine, it is not my fault.

As a climax to the whole, your concluding remark that the test of the measuring tape "you know to be fallacious," warns me that further remarks are needless.

I am, Sir,

Obediently yours,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 30th Nov., 1860.

SIR,

I wrote you from Harrogate acknowledging the receipt of your favour of the 21st Sept. You are therefore aware, that absence from home has prevented me from replying to it at an earlier date.

The tone and tenor of the whole letter warn me, that in your opinion it is time to bring our correspondence to a close; and the concluding paragraphs of it, I take to be

an intimation that it is your intention to act upon this opinion. You can hardly suppose me to have carefully read your letter, without arriving at a similar conclusion. If the correspondence be continued further, all I can say is, it will be traceable to an act of yours, not mine. You will of course now understand that I write under the conviction, that I shall receive no further communication from you on this subject. Under these circumstances, I shall not confine myself to the ordinary limits of a letter. I purpose to introduce a variety of demonstrations, and illustrations, which I consider necessary, to fully elucidate my theory; previously to publishing the entire correspondence.

Assuming you to be sincere, and giving due weight to the various remarks in your latter communications, it would argue presumption in me to ask, and folly to expect, that you should give consideration to any further arguments I may have to advance, on the question at issue between us. You may, or you may not read, what I am about to write. In either case, it is immaterial to me. I now write to you, but not exclusively for you.

"The Quadrature of the Circle" is a subject, the importance of which no Mathematician will dispute. You have induced the correspondence between us on this great question, and I sincerely thank you for it. To me it has been both interesting and instructive. I have no fear that we shall have readers, (ultimately, if not immediately,) who will examine the facts and arguments on both sides, and will impartially judge between us, and so far my purpose is served.

" *Magna est veritas, et prevalebit.*"

Had I been guilty of making the unqualified asser-

tion, with which you charge me, in that paragraph of your letter in which you say, "*Courtesy prevents my making the same remarks upon your reasoning, which you have made upon that of every Mathematician in Europe, when you say that we 'maintain a theory so utterly absurd, as that there is no definable relation between the diameter and circumference of a circle.'*" I could understand, that self-respect would have prevented you from using such language as would fitly describe, so gross a piece of impertinence and folly on my part; but you must excuse me for declining to give you credit for the courtesy to which you lay claim, in hesitating to give utterance to the impressions of your own mind, on the fact of my daring to differ from your "*authoritative dictation*," and declining to accept your assumptions and those of the other Mathematicians of Europe.

It is just possible you may have read thus far. If so, justice to yourself may induce you to refer to, and to read carefully, my letter of the 13th Sept. Supposing you to do so, I would ask you to examine the facts, to which I directed your attention. I would request you to read the whole paragraph, from which you have made a very imperfect extract. I would entreat you to reflect, whether the paragraph will bear the construction you have been pleased to put upon it. And having done so, pray ask yourself the question:—Have I not been guilty of unfairly representing my opponent's statements? I dare venture to accept the answer of your own conscience.

You are a Mathematician, and assume to be an "authority," on this important question. As such, you undertook the self-imposed task, of proving that the reasoning in my pamphlet was based on a fallacy. You have failed to do so, and have resorted to making comments upon

unfair quotations from my letters. This you have done, either ignorantly, or wilfully. I care not which. Be it the one, or be it the other, such a course of procedure, is entirely subversive of the "authority" to which you lay claim on this great question.

You say, "*Humility, one of the first of qualifications in those who earnestly seek truth, is obviously not one of my characteristics.*" It is true, I am not, and never pretended to be, possessed of that species of humility, which is ready to accept anything on the mere authority of a great name; or, if after careful examination, opposed to the honest convictions of my own mind. This you might have discovered when you read my pamphlet. You must have known it, or fancied you knew it, when in an early stage of our correspondence, you professed to "*recognise me as a bold enquirer after truth, not deterred by the authoritative dictation of others;*" and it is rather late now, to adduce this as a charge against me. It is altogether beside the question in dispute, and I may pass it by without further comment.

You next proceed to remark:—"*All the conclusions you have arrived at, are based on the assumption that the 'Writer's Hypothesis' is correct. No one will dispute, that if the Circle you imagine can really be drawn, various definable relations will exist. I have so often told you that no such Circle can be drawn, that I feel it is childish to go on repeating the same thing.*"

The circle I "imagine," is a plane figure, enclosed by one line, which is called the circumference of the circle, it is such, that all straight lines drawn from a certain point within the figure to the circumference, are equal to one another. This point within the figure is called the centre of the circle. A straight line drawn through the

centre, and terminated both ways by the circumference, is called a diameter of the circle.

Do you really mean to deny, that such a circle can be drawn? Your language literally construed would imply as much. If so, it would appear we have never been agreed on first principles.

Such, however, is the definition of the circle to which I have directed my attention. That it is the true construction of a circle, I presume you will not venture to dispute. I have examined the properties of this circle on a system, never, I have reason to believe, before adopted by any man; and having done so, I have dared to doubt the assumptions of the Mathematician with reference to it.

Is there, or is there not, a definable relation between the diameter and circumference of a circle? This is the question at issue.

The modern Mathematician, firmly adhering to the philosophy of the ancients, and taking for granted what he has been taught, never appears to have attempted to examine the properties of a circle, except from a single point of view; and thus, failing to discover the commensurable relation between the diameter and circumference of a circle, boldly asserts, that no such relation does or can exist.

I on the contrary, have declined to take for granted what has so long been taught, and have examined the properties of a circle for myself, and in doing so, have adopted the more modern philosophy of Bacon; and by these means have arrived at a conclusion, that there is a definable relation between the diameter and circumference of a circle.

On the Baconian principle of induction, I sought for,

and have discovered, a number of undeniable facts, inherent in the very nature of a circle. To some of these facts, I have directed attention in my pamphlet, to others in our correspondence. Not one of these facts, have you ever ventured to dispute, or attempted to subvert; and you admit, if your words have any meaning at all, that the definable relation between the diameter and circumference of a circle which I have assumed, is in perfect harmony with all these facts, when you say—"*no one will dispute that if the circle you imagine can really be drawn, various definable relations will exist.*" Under these circumstances, I am justified on the principles of the Baconian philosophy, in maintaining that my data, though only hypothetical, represents the true relation of diameter to circumference in a circle.

You may deny, on the one hand, my right to pursue the inquiry into this important question, on any such principle; or, you may maintain, on the other hand, (as I presume you do) that the Mathematician has demonstrated in a manner so plain and obvious, that no definable relation does or can exist, between the diameter and circumference of a circle, that any further enquiry into the subject is a waste of time.

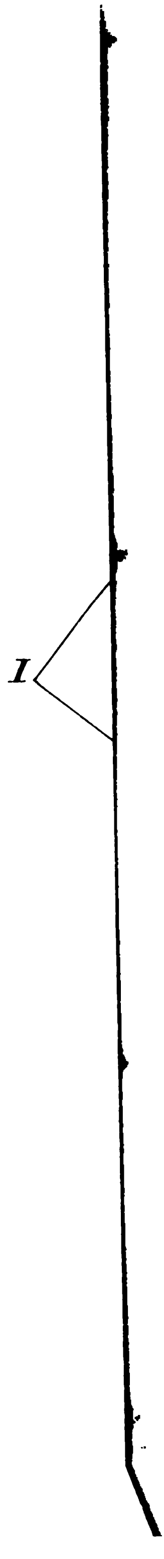
In the former case, you certainly do your utmost to undermine the very foundations of all modern philosophy. Is not the foundation on which every modern science is erected, to obtain in the first place a sufficient number of facts, and if after careful examination of these facts, they are found to be in perfect harmony with each other, and all thoroughly consistent with certain hypothetical data, then, to adopt such data as an axiom in respect to the particular science? This you cannot deny, and I may tell you, that if you make this a charge against me,

you attack the authority of all the wisest philosophers of modern times, from Bacon downwards.

In the latter case, I deny that the supposed demonstration of the Mathematician is plain and obvious; and maintain on the contrary, that it is based on an assumption, without the slightest proof, and is not true in fact; and this I shall demonstrate before I conclude this letter.

In the mean time, however, I shall make some further remarks on your assertions, and direct attention to some further interesting illustrations, in elucidation of my theory. You make the assertion that "*all the conclusions I have arrived at, are based on the assumption that the 'writer's hypothesis is correct.'*"

This I deny. It is true, that I have directed your attention to one class of facts, with reference to a circle in combination with other geometrical figures, which, "on the assumption that the writer's hypothesis is correct," I have shewn to have relations to each other, and with the circle, all of which relations can be arithmetically expressed in finite terms. But, I have also directed your attention to another class of facts, having no reference either to the relation of diameter to circumference in a circle, or to the writer's hypothesis. I have shewn that a variety of perfect geometrical figures may exist in combination, all derived from a circle, each having its boundaries defined by the circle, not one of which could be described without the circle, and all having perfect relations to each other, which can be arithmetically expressed in finite terms; and the only conclusion I have drawn from this class of facts is, that this would be an utter impossibility if there were no definable relation between the diameter and circumference of a circle. It is therefore a mistake on your part to say that "*all*" my conclusions are based


I

on the assumption that the "writer's hypothesis is correct."

This latter class of facts I consider of the utmost importance in the consideration of this question, I shall therefore direct attention to some further illustrations of the same character.

Let me refer you to the enclosed diagram, No. 1, (see Fig. X.) and the following construction of it.

With any diameter, say the line A B, describe the circle X. Inscribe the equilateral triangle A C D; and on C D a side of it, describe the square C E F D. With A B as perpendicular, describe the right-angled triangle A B G; making G B equal to three-fourths of A B. With A B as hypotheneuse, describe the right-angled triangle A H B; making A H equal to four-fifths, and B H equal to three-fifths of A B. On A G, the hypotheneuse of the triangle A B G, describe the square A G I K.

The diameter of the circle X may be any given number, say 8. Then, the area of a square circumscribing the circle X, will be $8^2 = 64$; and the area of a square inscribed in circle X, will be 32. The arithmetical mean between these two numbers is, $\frac{1}{2}(64 + 32) = 48$; and $\sqrt{48}$ will be the value of C D, a side of the equilateral triangle A C D; and, therefore, the area of the square C E F D will be 48.

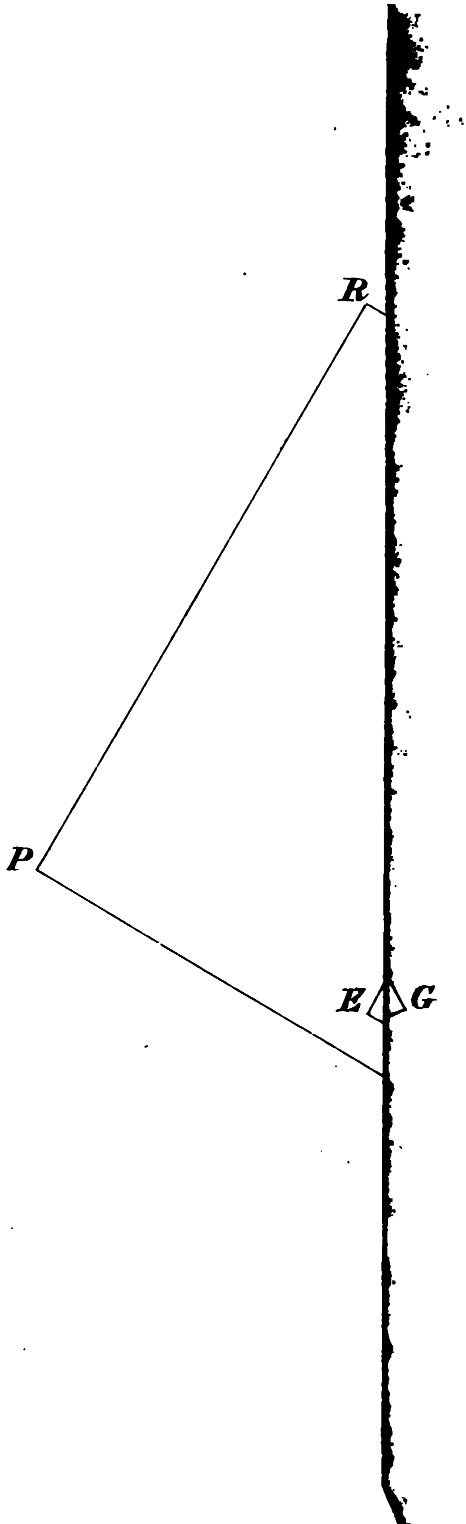
The side G B is to the side A B, in the triangle A B G, in the ratio of 3 to 4, by construction. Therefore $\frac{3}{4}(8)$, $= \cdot 75 \times 8 = 6$, will be the value of the side G B. The given value of the side A B, in the triangle A B G is 8. And $\frac{1}{2}(6 \times 8) = 24$, will be the value of the area of the triangle A B G; and is equal to half the area of the square C E F D, described on C D, a side of the equilateral triangle A C D. For, $24 \times 2 = 48$, is the area of the square C E F D.

Again, the side A H is to the side A B, in the triangle A H B, in the ratio of 4 to 5; and the side B H to the side A B, in the ratio of 3 to 5; by construction. Therefore, $\frac{4}{5}(8), = \cdot 8 \times 8, = 6 \cdot 4$, will be the value of the side A H; and $\frac{3}{5}(8), = \cdot 6 \times 8, = 4 \cdot 8$, will be the value of the side B H, of the triangle A H B; and $\frac{1}{2}(6 \cdot 4 \times 4 \cdot 8), = 15 \cdot 36$, will be the value of the area of the triangle A H B; and is equal to the three and one-eighth part of the area of the square C E F D, described on C D, a side of the inscribed equilateral triangle A C D. For, $15 \cdot 36 \times 3 \cdot 125 = 48$, is the area of the square C E F D. It will be observed that A B, the diameter of the Circle X, is common to the two triangles A B G, and A H B.

Again, the side A G of the triangle A B G, is also a side of the square A G I K, and is equal to $\sqrt{AB^2 + GB^2} = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$. Therefore, $10^2 = 100$, will be the area of the square A G I K. But, the area of a square inscribed in the circle X, is 32. Therefore, the area of square A G I K, is equal to three and one-eighth times the area of a square inscribed in the circle X. For, $32 \times 3 \cdot 125 = 100$, is the area of square A G I K.

Again, the area of square A G I K, is to the area of the triangle A B G, as the area of a square circumscribed about the circle X, to the area of the triangle A H B. For, the area of square A G I K is 100. The area of the triangle A B G is 24. The area of a square circumscribed about the circle X is 64. The area of the triangle A H B is 15·36. And as $100 : 24 :: 64 : 15 \cdot 36$.

Again, if a regular hexagon be inscribed within the circle X, each side of the hexagon will be equal to a radius of the circle = 4. Therefore, $4 \times 6 = 24$, will be the value of the perimeter of the hexagon; and is exactly equal to the perimeter of the triangle A B G. For, 6 +


R
P
E
G

$8 + 10 = 24$, is the value of the perimeter of the triangle A B G.

Again, if a regular dodecagon be inscribed in the circle X, the area of the dodecagon will be exactly equal to the area of the square C E F D, described on C D, a side of the inscribed equilateral triangle A C D; as I have demonstrated in my letter of the 7th September. Therefore, definable relations exist between the triangles A B G, A H B, and the dodecagon, exactly similar to those existing between the triangles A B G, A H B, and the square C E F D; viz., The area of the dodecagon will be exactly equal to twice the area of the triangle A B G, and three and one-eight times the area of the triangle A H B.

All the geometrical figures here referred to, are connected with the circle, are brought into combination by means of the circle, and have relations to each other in a variety of ways, which can be expressed in finite terms; and it would be strange indeed, if such relations could exist among them and be so expressed, and there be no definable relation between the diameter and circumference of a circle, the figure by means of which, they are all generated.

I shall now direct attention to another combination of geometrical figures, having definable relations to each other, a diagram of which may be constructed by means of a circle, without any reference to the relation between the diameter and circumference of a circle, or to the "writer's hypothesis."

Permit me to refer you to the diagram No. 2, (see Fig. XI.) and the following construction of it.

From a point O as centre, and with any radius, describe the circle X. Draw the diameter A B. With the point B as centre and B O as radius, describe the circle Y. From the point B draw B C, B D, at right-angles to A B, making B C, B D, equal to B O, a radius of the circles

X and Y, and common to both. Join A C, A D, describing the right-angled triangles A B C, A B D. On A C, A D, describe the squares A C E F, A D G H. Join F, H, and on F H describe the square F I K H. In the square F I K H, inscribe the circle Z. In the circle Z inscribe the equilateral triangle L M N. On L M, a side of the equilateral triangle L M N, describe the square L M P R. In the generating circle X, inscribe the equilateral triangle A S T.

The diameter of the generating circle X may be any given number, say 8.

Then, A B, the diameter of circle X, is common to the two right-angled triangles A B C, A B D, and the sides B C, B D, of the triangles A B C, A B D, are each equal to the radius of the circle X, by construction. Therefore, $AB^2 + BC^2$ or $BD^2 = 8^2 + 4^2, = 64 + 16 = 80$; and $\sqrt{80}$ will be the value of A C and A D, the third sides of the triangles A B C and A B D. But, A C and A D are also sides of the squares A C E F and A D G H, and the value of the area of each of these squares, must therefore be $\sqrt{80}^2 = 80$. And, $\frac{1}{2}$(A B $\times$ B C), or, $\frac{1}{2}$ (A B $\times$ B D), $= \frac{1}{2}(8 \times 4), = 16$, must be the value of the area of each of the triangles A B C, and A B D.

The diameter of circle Z, is equal to twice the diameter of the generating circle X, $= 16$. Therefore, $16^2 = 256$, will be the area of square F I K H, circumscribed about the circle Z; and the area of a square inscribed in circle Z, must be $\frac{1}{2}(256) = 128$; and the arithmetical mean between these two numbers is, $= \frac{1}{2}(256 + 128), = 192$; and $\sqrt{192}$ will be the value of L M, a side of the equilateral triangle L M N, inscribed in the circle Z; and the area of the square L M P R, described on L M, must be $\sqrt{192}^2 = 192$.

But, the area of square A C E F, plus the area of square A D G H, plus the area of triangle A B C, plus the area of triangle A B D, equal to 80 + 80 + 16 + 16 = 192; and is exactly equal to the area of the square L M P R, described on L M, a side of the equilateral triangle L M N, inscribed in the circle Z.*

If the diameter of the generating circle X, be an incommensurable quantity, it is true, we can only arrive at an approximative value of some of these geometrical figures, but this approximation may be carried as close as we please.

For example: Let A B, the diameter of circle X, be $\sqrt{2}$.

Then, $AB^2 + BC^2$ or BD^2, $= \overline{\sqrt{2}}^{2} + \overline{\tfrac{1}{2}(\sqrt{2})}^{2}, = \overline{\sqrt{2}}^{2} + \overline{\sqrt{\tfrac{1}{4} \times 2}}^{2}, = \overline{\sqrt{2}}^{2} + \overline{\sqrt{\cdot25 \times 2}}^{2}, = \overline{\sqrt{2}}^{2} + \overline{\sqrt{\cdot5}}^{2}, = \sqrt{2 + \cdot5}, = \sqrt{2\cdot5}$; and $\sqrt{2\cdot5}$ will be the value of A C, and A D; and A C, A D, are sides of the squares A C E F, and A D G H, and the value of the area of each of these squares must be $\overline{\sqrt{2\cdot5}}^{2} = 2\cdot5$.

Then, A B, the diameter of the generating circle X, is common to the right-angled triangles A B C, and A B D, and the sides B C, B D, of these triangles are equal to the radius of the generating circle. Therefore, $\tfrac{1}{2}\{\sqrt{2} \times \tfrac{1}{2}(\sqrt{2})\}, = \tfrac{1}{2}\{\sqrt{2} \times (\sqrt{\tfrac{1}{4} \times 2})\}, = \tfrac{1}{2}\{\sqrt{2} \times (\sqrt{\cdot25 \times 2})\}, = \tfrac{1}{2}(\sqrt{2} \times \sqrt{\cdot5}), = \tfrac{1}{2}(1\cdot41421356\ \&c. \times \cdot70710678\ \&c.), = \tfrac{1}{2}(\cdot9999999966439368\ \&c.), = \cdot4999999983219684\ \&c.$, will

*** It may be observed, that I have omitted to direct attention to the fact, that the two triangles A B C, and A B D, together form the isoseles triangle A C D, of which the value of the area is 32. Therefore, the area of the triangle A C D, is exactly equal to the area of a square inscribed in the generating circle X.**

be the approximative value of the area of each of the triangles A B C, and A B D.

The diameter of Circle Z, is equal to twice the diameter of the generating Circle X, $= 2\,(\sqrt{2}) = \sqrt{2^2 \times 2} = \sqrt{4 \times 2} = \sqrt{8}.$ Therefore, the area of the square F I K H, circumscribed about the Circle Z, will be 8; and the area of a square inscribed in the Circle Z will be 4; and the arithmetical mean between these two numbers is, $\frac{1}{2}(8 + 4) = 6$; and $\sqrt{6}$ will be the value of L M, a side of the equilateral triangle L M N, inscribed in the Circle Z; and the area of the square L M P R, described on L M, must be $\sqrt{6}^2 = 6$. But, the area of square A C E F, plus the area of square A D G H, plus the area of triangle A B C, plus the area of triangle A B D, $= 2{\cdot}5 + 2{\cdot}5 + {\cdot}49999999832219684$ &c. $+ {\cdot}49999999832219684$ &c. $= 5{\cdot}99999999664439368$ &c; and is a close approximation to the area of square L M P R, described on L M, a side of the inscribed equilateral triangle L M N; and the approximation may be made as close as we please, by extending the number of decimals.

Again, it will be observed that the generating Circle X, is an inscribed circle to the equilateral triangle L M N, and a circumscribed circle to the equilateral triangle A S T, and the area of the triangle L M N, is equal to four times the area of the triangle A S T.

Thus, we have again a variety of geometrical figures, having commensurable relations between each other. They are all constructed by means of a circle, could not be constructed independantly of the circle, and all exist in harmonious combination with the circle. All these facts I have demonstrated without the slightest reference to the "writer's hypothesis," or to the relation between the diameter and circumference of a circle.

At this point I might have reiterated the whole paragraph of my last letter, from which you have made the imperfect extract, and then commented on it; and I could have done so without the least idea of being uncourteous, or even dreaming of such a thing, as being thought uncourteous. After your remarks, it is perhaps better I should not repeat the question there put, more particularly, as you have the paragraph to refer to, if you should think it worth your while to do so.

I have subjected my theory to every conceivable test, both mathematical and mechanical, with an honest determination to find a flaw if possible; and having failed to do so, I now unhesitatingly propound it, as the true theory on this important question.

If it be false, it can surely be proved to be so, by that learned fraternity who claim to be regarded as our guides on this subject; and it is their duty to afford such proof. If it be true, it is equally their duty to acknowledge it. If they do their duty, one or other of these courses they must adopt. In propounding the theory, I have done my duty; and I now call upon them to do theirs.

The definition of my theory, may be expressed in the following terms:—

IN EVERY CIRCLE, THE CIRCUMFERENCE IS EXACTLY EQUAL TO THREE AND ONE-EIGHTH TIMES ITS DIAMETER; OR, EXPRESSED IN EQUIVALENT TERMS—IN EVERY CIRCLE, EIGHT CIRCUMFERENCES ARE EXACTLY EQUAL TO TWENTY-FIVE DIAMETERS. THEREFORE, IF THE DIAMETER OF A CIRCLE BE 1, THE CIRCUMFERENCE IS 3·125; AND IN EVERY CIRCLE, THE RATIO OF DIAMETER TO CIRCUMFERENCE IS, AS 1 TO 3·125.

THE AREA OF ANY CIRCLE, IS TO THE AREA OF A SQUARE DESCRIBED ON ITS DIAMETER, AS THE CIRCUMFERENCE OF THE CIRCLE, TO THE PERIMETER OF THE SQUARE. THEREFORE, IF THE DIAMETER OF A CIRCLE BE 1, THE PERIMETER OF A SQUARE, DESCRIBED ON ITS DIAMETER WILL BE 4; AND AS 4 : 3·125 : : 1 : ·78125; AND THE AREA OF EVERY SQUARE, IS TO THE AREA OF AN INSCRIBED CIRCLE, IN THE RATIO OF 1 TO ·78125. HENCE:—THE AREA OF EVERY CIRCLE, IS EXACTLY EQUAL TO THREE AND ONE-EIGHTH TIMES THE AREA OF A SQUARE, DESCRIBED ON ITS RADIUS.

I shall now proceed to give some further demonstrations, and illustrations, in support of this theory.

The enclosed diagram No. 3, (see Fig. XII.) represents two squares, one circumscribed about, and the other inscribed in a circle It is not necessary I should demonstrate, that the area of the circumscribed square is equal to twice the area of the inscribed square. This I assume to be admitted, as I have never met with any one who thought of disputing it.

The area of the inscribed square may be any given number, say 7·3. Then, as 16 : 25 : : 7·3 : 11·40625, the area of the circle. As ·78125 : 1 : : 11·40625 : 14·6, the area of the circumscribed square; and is equal to twice the area of the inscribed square, which is its admitted value.

The area of the circumscribed square may be any given number, say 9·1. Then, as 1 : ·78125 : : 9·1 : 7·109375, the area of the circle. As 25 : 16 : : 7·109375 : 4·55, the area of the inscribed square; and is equal to half the area of the circumscribed square, which is its admitted value.

FIGURE XII.

(Diagram III.)

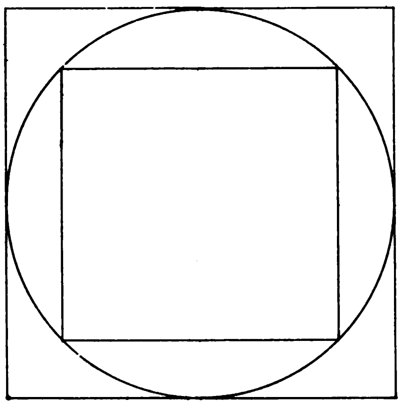


FIGURE XIII.

(Diagram IV.)

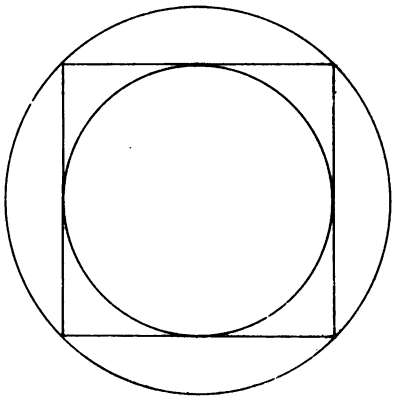
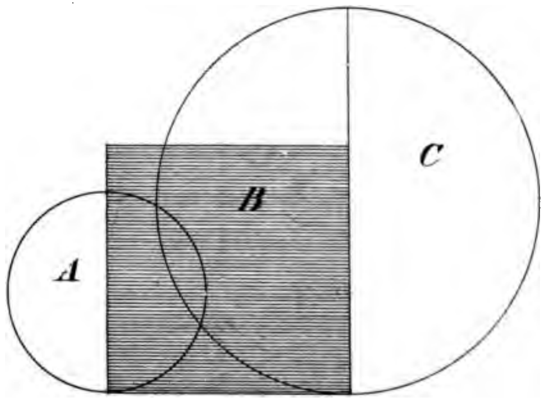


FIGURE XIV.

(Diagram V.)



The diagram No. 4 (see Fig. XIII.) represents two circles, one circumscribed about, and the other inscribed in a square. I again assume it to be admitted, that the area of the circumscribed circle, is equal to twice the area of the inscribed circle.

The area of the inscribed circle may be any given number, say 8·2. Then, as 16 : 25 : : 8·2 : 12·815, the area of the square. As ·78125 : 1 : : 12·8125 : 16·4, the area of the circumscribed circle; and is equal to twice the area of the inscribed circle, which is its admitted value.

The area of the circumscribed circle may be any given number, say 6·5. Then, as 25 : 16 : : 6·5 : 4·16, the area of the square. As 1 : ·78125 : : 4·16 : 3·25, the area of the inscribed circle; and is equal to half the area of the circumscribed circle, which is its admitted value.

In the diagram No. 5, (see Fig. XIV.) let the diameter of circle A, be equal to four-fifths of the side of square B; and the diameter of circle C, equal to eight-fifths of the side of square B. Then, the diameter of circle C, will be equal to twice the diameter of circle A; the circumference of circle C, will be equal to twice the circumference of circle A; and the area of circle C, will be equal to four times the area of circle A.

The diameter of circle A may be any given number, say $\sqrt{6{\cdot}3}$. Then, $\frac{5}{4}$ ($\sqrt{6{\cdot}3}$), = $\sqrt{\frac{25}{16} \times 6{\cdot}3}$, = $\sqrt{1{\cdot}5625 \times 6{\cdot}3}$, = $\sqrt{9{\cdot}84375}$, will be the value of the side of square B. $\frac{8}{5}$ ($\sqrt{9{\cdot}84375}$), = $\sqrt{\frac{64}{25} \times 9{\cdot}84375}$, = $\sqrt{2{\cdot}56 \times 9{\cdot}84375}$, = $\sqrt{25{\cdot}2}$, will be the value of the diameter of circle C; and is equal to twice the diameter of circle A. For, 2 ($\sqrt{6{\cdot}3}$), = $\sqrt{2^2 \times 6{\cdot}3}$, = $\sqrt{4 \times 6{\cdot}3}$, = $\sqrt{25{\cdot}2}$, the diameter of circle C.

Again, the circumference of Circle A may be any given number, say $\sqrt{6}$. Then, $\sqrt{6} \div 3{\cdot}125$, = $\sqrt{6} \div \frac{25}{8}$,

$= \sqrt{6 \div \frac{625}{64}}$, $= \sqrt{6 \div 9{\cdot}765625}$, $= \sqrt{{\cdot}6144}$, will be the value of the diameter of Circle A. $\frac{5}{4}$ $(\sqrt{{\cdot}6144})$, $= \sqrt{\frac{25}{16} \times {\cdot}6144}$, $= \sqrt{1{\cdot}5625 \times {\cdot}6144}$, $= \sqrt{{\cdot}96}$, will be the value of the side of square B. $\frac{8}{5}$ $(\sqrt{{\cdot}96})$, $= \sqrt{\frac{64}{25} \times {\cdot}96}$, $= \sqrt{2{\cdot}56 \times {\cdot}96}$, $= \sqrt{2{\cdot}4576}$, will be the value of the diameter of circle C. $3\frac{1}{8}(\sqrt{2{\cdot}4576})$, $= \frac{25}{8}(\sqrt{2{\cdot}4576})$, $= \sqrt{\frac{625}{64} \times 2{\cdot}4576}$, $= \sqrt{9{\cdot}765625 \times 2{\cdot}4576}$, $= \sqrt{24}$, will be the value of the circumference of circle C; and is equal to twice the circumference of circle A. For, $2\,(\sqrt{6})$, $= \sqrt{2^2 \times 6}$, $= \sqrt{4 \times 6}$, $= \sqrt{24}$, the circumference of circle C.

Again, the area of circle A may be any given number, say $7{\cdot}7$. Then, as $\cdot78125 : 1 :: 7{\cdot}7 : 9{\cdot}856$, the area of a square circumscribed about the circle; and $\sqrt{9{\cdot}856}$ will be the value of the diameter of circle A. $\frac{5}{4}$ $(\sqrt{9{\cdot}856})$, $= \sqrt{\frac{25}{16} \times 9{\cdot}856}$, $= \sqrt{1{\cdot}5625 \times 9{\cdot}856}$, $= \sqrt{15{\cdot}4}$, will be the value of the side of square B. $\frac{8}{5}$ $(\sqrt{15{\cdot}4})$, $= \sqrt{\frac{64}{25} \times 15{\cdot}4}$, $= \sqrt{2{\cdot}56 \times 15{\cdot}4}$, $= \sqrt{39{\cdot}424}$, will be the value of the diameter of circle C. $\sqrt{39{\cdot}424}^{\,2} = 39{\cdot}424$, will be the value of the area of a square circumscribed about the circle C; $39{\cdot}424 \times {\cdot}78125 = 30{\cdot}8$, will be the value of the area of circle C; and is equal to four times the area of circle A. For, $7{\cdot}7 \times 4 = 30{\cdot}8$, the area of circle C.

Again, Let the diameter of circle A be $7{\cdot}7$. Then, $7{\cdot}7^2 = 59{\cdot}29$, will be the value of the area of a square circumscribed about the circle; and $59{\cdot}29 \times {\cdot}78125 = 46{\cdot}3203125$, will be the value of the area of circle A. $\frac{5}{4}$ $(7{\cdot}7)$, $= 1{\cdot}25 \times 7{\cdot}7$, $= 9{\cdot}625$, will be the value of the side of square B. $9{\cdot}625^2 = 92{\cdot}640625$, will be the value of the area of square B; and is equal to twice the area of circle A. $\frac{8}{5}$ $(9{\cdot}625)$, $= 1{\cdot}6 \times 9{\cdot}625$, $= 15{\cdot}4$, will be the value of the diameter of circle C. $15{\cdot}4^2 = 237{\cdot}16$, will be the value of the area of a square circumscribed about the circle C. $237{\cdot}16 \times {\cdot}78125 = 185{\cdot}28125$, will be the

value of the area of circle C; and is equal to twice the area of square B, and four times the area of circle A.

Again, the diameter of circle C may be any given number, say $\sqrt{7}$. Then, $\frac{5}{8}(\sqrt{7}), = \sqrt{\frac{25}{64} \times 7}, = \sqrt{\cdot 390625 \times 7} = \sqrt{2\cdot 734375}$, will be the value of the side of square B. $\frac{4}{5}(\sqrt{2\cdot 734375}), = \sqrt{\frac{16}{25} \times 2\cdot 734375}, = \sqrt{\cdot 64 \times 2\cdot 734375}, = \sqrt{1\cdot 75}$, will be the value of the diameter of circle A; and is equal to half the diameter of circle C. For, 2 $(\sqrt{1\cdot 75}), = \sqrt{2^2 \times 1\cdot 75}, = \sqrt{4 \times 1\cdot 75}, = \sqrt{7}$, is the given value of the diameter of circle C.

Again, the circumference of circle C may be any given number, say $\sqrt{5}$. Then, $\sqrt{5} \div 3\cdot 125, = \sqrt{5 \div \frac{25}{8}}, = \sqrt{5 \div \frac{625}{64}}, = \sqrt{5 \div 9\cdot 765625}, = \sqrt{\cdot 512}$, will be the value of the diameter of circle C. $\frac{5}{8}(\sqrt{\cdot 512}), = \sqrt{\frac{25}{64} \times \cdot 512}, = \sqrt{\cdot 390625 \times \cdot 512}, = \sqrt{\cdot 2}$, will be the value of the side of square B. $\frac{4}{5}(\sqrt{\cdot 2}), = \sqrt{\frac{16}{25} \times \cdot 2}, = \sqrt{\cdot 64 \times \cdot 2} = \sqrt{\cdot 128}$, will be the value of the diameter of circle A. $3\frac{1}{8}(\sqrt{\cdot 128}), = \frac{25}{8}(\sqrt{\cdot 128}), = \sqrt{\frac{625}{64} \times \cdot 128}, = \sqrt{9\cdot 765625 \times \cdot 128}, = \sqrt{1\cdot 25}$, will be the value of the circumference of circle A; and is equal to half the circumference of circle C. For, 2 $(\sqrt{1\cdot 25}), = \sqrt{2^2 \times 1\cdot 25}, = \sqrt{4 \times 1\cdot 25}, = \sqrt{5}$, is the given value of the circumference of circle C.

Again, the area of circle C may be any given number, say 9. Then, as $\cdot 78125 : 1 :: 9 : 11\cdot 52$, the area of a square circumscribed about the circle; and $\sqrt{11\cdot 52}$ will be the value of the diameter of circle C. $\frac{5}{8}(\sqrt{11\cdot 52}), = \sqrt{\frac{25}{64} \times 11\cdot 52}, = \sqrt{\cdot 390625 \times 11\cdot 52}, = \sqrt{4\cdot 5}$, will be the value of the side of square B. $\frac{4}{5}(\sqrt{4\cdot 5}), = \sqrt{\frac{16}{25} \times 4\cdot 5}, = \sqrt{\cdot 64 \times 4\cdot 5}, = \sqrt{2\cdot 88}$, will be the value of the diameter of circle A. $\sqrt{2\cdot 88}^2 = 2\cdot 88$, will be the value of the area of a square circumscribed about the circle A; and $2\cdot 88 \times \cdot 781215 = 2\cdot 25$, will be the value of the area of circle A; and is equal to one-fourth of the area of circle C. For, $2\cdot 25 \times 4 = 9$, is the given value of the area of circle C.

Again, Let the diameter of circle C be 9. Then, $9^2 = 81$, will be the value of the area of a square circumscribed about the circle; and $81 \times \cdot78125 = 63\cdot28125$, will be the value of the area of circle C. $\frac{5}{8}(9), = \cdot625 \times 9, = 5\cdot625$, will be the value of the side of square B. $5\cdot625^2 = 31\cdot640625$, will be the value of the area of square B; and is equal to half the area of circle C. $\frac{4}{5}(5\cdot625), = \cdot8 \times 5\cdot625, = 4\cdot5$, will be the value of the diameter of circle A. $4\cdot5^2 = 20\cdot25$, will be the value of the area of a square circumscribed about the circle; and $20\cdot25 \times \cdot78125 = 15\cdot8203125$, will be the value of the area of circle A; and is equal to half the area of square B, and one-fourth the area of circle C.

In this diagram we have the two circles A and C, the diameter of circle C being equal to twice the diameter of circle A, by construction; and the square B intermediate between the two, and having by construction, commensurable relations with the diameters of both. I have demonstrated that the value of the diameter, circumference, or area of either circle, may be the given quantity; and the admitted value of each of these in the other circle may be arrived at, through the intervention of the intermediate square.

Your reasoning in various parts of our correspondence would imply, that in your opinion a variety of data may be assumed, by means of which commensurable relations between a circle and other geometrical figures can be produced. Now, if you can shew me any other data, by which you can produce the results I have demonstrated in this diagram, I shall be prepared to give up the contest.

In the next place, let it be required to describe a square, equal in area to a given circle.

FIGURE XV.

(Diagram VI.)

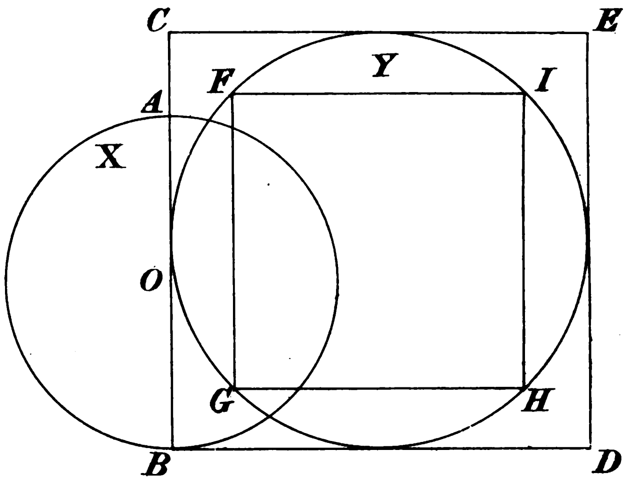
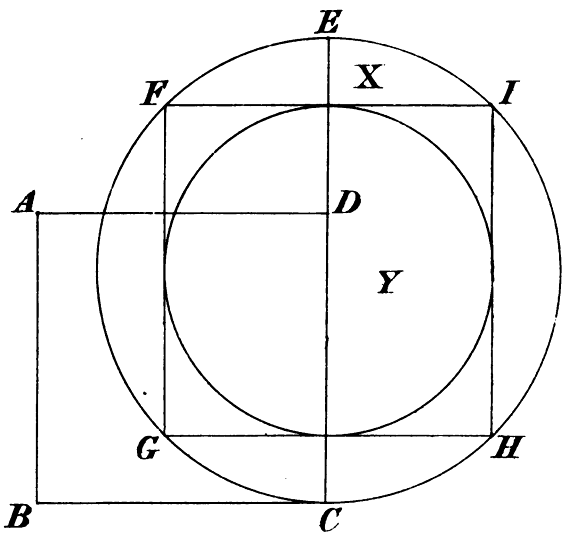


FIGURE XVI.

(Diagram VII.)



Permit me to direct your attention to the construction of the enclosed diagram No. 6, (see Fig. XV.)

With the point O as centre, and with any radius, describe the circle X. Draw the diameter A B. Produce A B to C, making B C equal to five-fourths of A B. On C B, describe the square C B D E. In the square C B D E, inscribe the circle Y, and in the circle Y, inscribe the square F G H I. Then, the area of the square F G H I, is equal to the area of circle X.

The area of circle X may be any given number, say 63. Then, as ·78125 : 1 : : 63 : 80·64, the area of a circumscribed square; and $\sqrt{80·64}$ will be the value of the diameter of circle X. $\frac{5}{4}$ ($\sqrt{80·64}$), $= \sqrt{\frac{25}{16} \times 80·64}, = \sqrt{1·5625 \times 80·64}, = \sqrt{126}$, will be the value of the side of square C B D E. $\sqrt{126}^2 = 126$, will be the area of square C B D E. As 1 : ·78125 : : 126 : 98·4375, the area of circle Y. As 25 : 16 : : 98·4375 : 63, the area of square F G H I; and is equal to the given area of circle X.

Again, let it be required to describe a circle equal in area to a given square. For the construction of the required diagram, let me refer to the enclosed diagram No. 7, (see Fig. XVI.) to which I now beg to direct your attention.

On the line A B, describe the square A B C D. Produce C D to E, making C E equal to eight-fifths of C D. With C E as diameter, describe the circle X. In the circle X, inscribe the square F G H I; and in the square F G H I, inscribe the circle Y. Then, the area of circle Y is equal to the area of the square A B C D.

The area of square A B C D may be any given number, say 47. Then, $\sqrt{47}$ will be the value of the side of the square A B C D. $\frac{8}{5}$ ($\sqrt{47}$), $= \sqrt{\frac{64}{25} \times 47}, = \sqrt{2·56 \times 47}, = \sqrt{120·32}$, will be the value of the diameter of circle X.

$\sqrt{120{\cdot}32}^2 = 120{\cdot}32$, will be the area of a square described on the diameter of circle X; and $120{\cdot}32 \times {\cdot}78125 = 94$, will be the area of circle X. As 25 : 16 : : 94 : 60·16, the area of the inscribed square F G H I; and $60{\cdot}16 \times {\cdot}78125 = 47$, will be the area of circle Y; and is equal to the given area of the square A B C D.

In the enclosed diagram No. 8, (see Fig. XVII.) there is described a combination of perfect geometrical figures, by means of which the various relations of this particular description are demonstrated, and to it I would now direct attention.

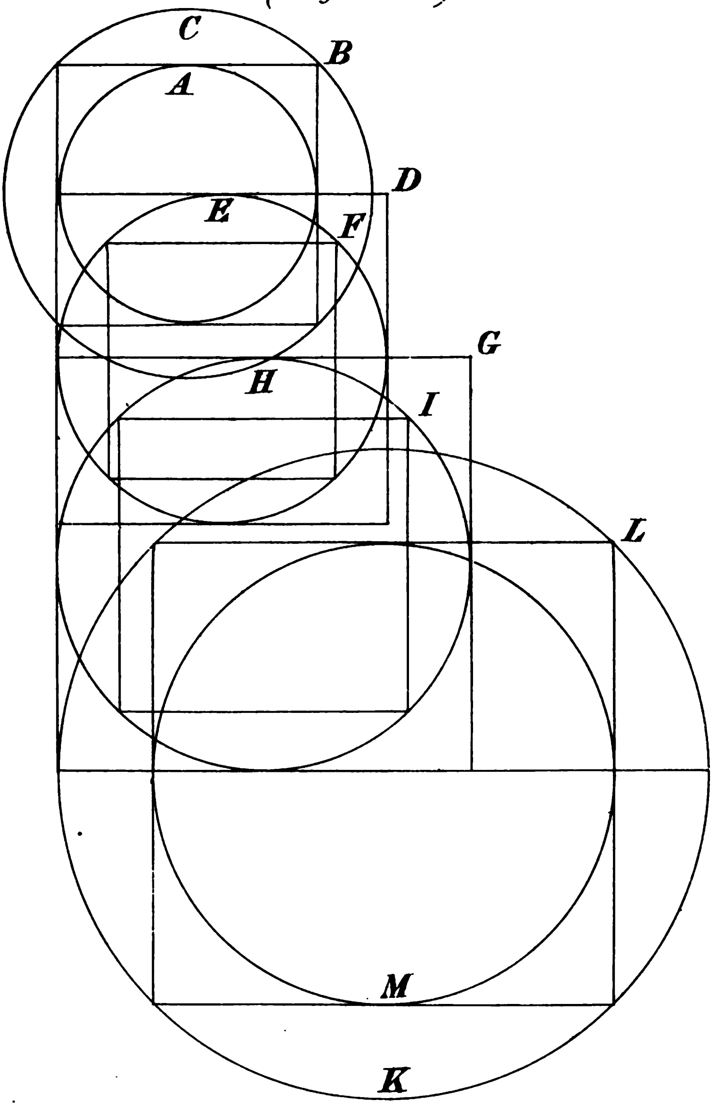
Let the diameter of circle A be 1·28. Then, $1{\cdot}28^2 = 1{\cdot}6384$, will be the area of a square described on the diameter of circle A; and $1{\cdot}6384 \times {\cdot}78125 = 1{\cdot}28$, will be the area of circle A; and the values of the diameter and area of circle A, are represented by the same arithmetical symbols.

Then, as ·78125 : 1 : : 1·28 the area of circle A, to 1·6384 the area of the circumscribed square B. As 16 : 25 : : 1·6384 the area of square B, to 2·56 the area of the circumscribed circle C; and the area of circle C is equal to twice the area of the generating circle A.

Let the side of square D be equal to five-fourths of the diameter of circle A. Then, $\frac{5}{4}(1{\cdot}28), = 1{\cdot}25 \times 1{\cdot}28, = 1{\cdot}6$, will be the value of the side of square D; and $1{\cdot}6^2 = 2{\cdot}56$, will be the area of square D; and the area of square D is equal to the area of circle C, and twice the area of circle A. Then, the area of square D is 2·56, and $2{\cdot}56 \times {\cdot}78125 = 2$, will be the area of circle E, inscribed in square D. As 25 : 16 : : 2 the area of circle E, to 1·28 the area of the inscribed square F; and the area of square F, is equal to the area of the generating circle A.

FIGURE XVII.

(Diagram VIII.)



Then, $\sqrt{2 \cdot 56} = 1 \cdot 6$, is the value of the side of square D, and diameter of circle E. Let the side of square G be equal to five-fourths of the side of square D, and diameter of circle E. Then $\frac{5}{4}$ $(1 \cdot 6)$, $= 1 \cdot 25 \times 1 \cdot 6$, $= 2$, will be the value of the side of square G; and $2^2 = 4$, will be the area of square G; and the area of square G, is equal to twice the area of circle E.

Then, the area of square G is 4, and $4 \times \cdot 78125 = 3 \cdot 125$, will be the area of the inscribed circle H. As 25 : 16 : : 3·125 the area of circle H, to 2 the area of the inscribed square I; and the area of square I is equal to the area of circle E.

Then $\sqrt{4} = 2$, is the value of the side of square G, and diameter of circle H. Let the diameter of circle K be equal to eight-fifths of the side of square G, and diameter of circle H. Then $\frac{8}{5}$ (2), $= 1 \cdot 6 \times 2$, $= 3 \cdot 2$, will be the diameter of circle K. $3 \cdot 2^2 = 10 \cdot 24$, will be the area of a square described on the diameter of circle K; and $10 \cdot 24 \times \cdot 78125 = 8$, will be the area of circle K; and is equal to twice the area of square G. As 25 : 16 : : 8 the area of circle K, to 5·12 the area of the inscribed square L; and $5 \cdot 12 \times \cdot 78125 = 4$, will be the area of the inscribed circle M; and the area of circle M, is equal to the area of square G; and so we might proceed, *ad infinitum.*

Previous to directing attention to another diagram, I must refer to some facts given in the paper read by me in the Mathematical section of the "British Association for the Advancement of Science," at Aberdeen, and incidentally referred to in an early part of our correspondence, viz.: Either on a true or false value of the area of a circle of which the diameter is unity, it may be demonstrated, that the ratio that exists between the area of any square, and the area of an inscribed circle, exists also between the

perimeter of any square and the circumference of an inscribed circle; between the area of a circle and the area of a square, of which the perimeter is equal to the circumference of the circle; and between the diameter of a circle and the side of a square, of which the perimeter is equal to the circumference of the circle; and on this, the true theory of the value of a circle, the following results necessarily ensue:—

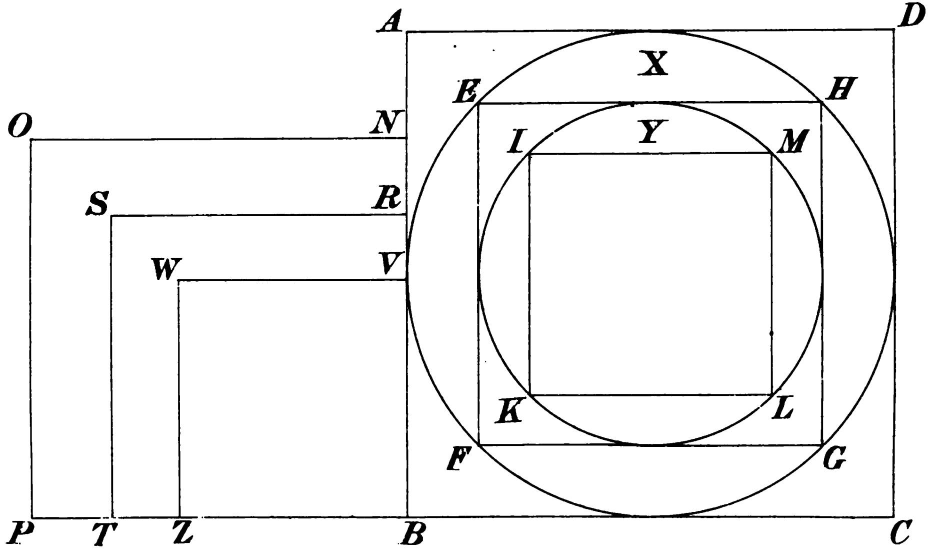
The area of any circle, is equal to $\frac{25}{32}$ parts of the area of a square described on its diameter. The circumference of every circle, is equal to $\frac{25}{32}$ parts of the perimeter of a square described on its diameter. The area of any square is equal to $\frac{25}{32}$ parts of the area of a circle, of which the circumference is equal to the perimeter of the square. The side of any square, is equal to $\frac{25}{32}$ parts of the diameter of a circle, when the perimeter of the square and circumference of the circle are equal.

In the enclosed diagram, No, 9, (see Fig. XVIII.) all the figures are generated from a given straight line A B, and to the construction of it I now direct attention.

On the straight line A B, describe the square ABCD, and in it inscribe the circle X. In the circle X, inscribe the square E F G H. In the square E F G H, inscribe the circle Y; and in the circle Y, inscribe the square I K L M. From A B, the given straight line, and a side of the square A B C D, cut off a part N B, equal to $\frac{25}{32}$ parts of A B; and on N B describe the square N O P B. From N B, a side of the square N O P B, cut off a part R B, equal to four-fifths of N B; and on R B describe the square RSTB. From R B, a side of the square R S T B, cut off a part V B, making V B equal to $\frac{25}{32}$ parts of the diameter of circle Y, or side of square E F G H; and on V B describe the square V W Z B.

FIGURE XVIII.

(Diagram IX)



The side A B of the square A B C D, the generating line of the diagram, may be any given number, say 2.

Then, $2^2 = 4$, will be the area of square A B C D, and $4 \times {\cdot}78125 = 3{\cdot}125$, will be the area of circle X. But, $\frac{25}{32}$ parts of the area of square A B C D, equal to $\frac{25}{32}$ (4), is also equal to the area of circle X. For $\frac{25}{32}(4), = \frac{25}{32} \times 4, = {\cdot}78125 \times 4, = 3{\cdot}125$, is the area of circle X.

The side N B of the square N O P B is, by construction, equal to $\frac{25}{32}$ parts of A B. Therefore, $\frac{25}{32}(2), = \frac{25}{32} \times 2, = {\cdot}78125 \times 2, = 1{\cdot}5625$, will be the value of the side N B, of the square N O P B; and $1{\cdot}5625^2 = 2{\cdot}44140625$, will be the area of square N O P B. But, $\frac{25}{32}$ parts of the area of circle X, equal to $\frac{25}{32}$ (3·125), is also equal to the area of square N O P B. For, $\frac{25}{32}(3{\cdot}125), = \frac{25}{32} \times 3{\cdot}125, = {\cdot}78125 \times 3{\cdot}125, = 2{\cdot}44140625$, is the area of square N O P B; and the area of circle X, is the mean proportional, between the area of square A B C D, and the area of square N O P B. For, $\sqrt{4 \times 2{\cdot}44140625}, = \sqrt{9{\cdot}765625}, = 3{\cdot}125$, is the area of circle X; and is also the mean proportional between the areas of the two squares A B C D, and N O P B.

The diameter of circle X, is equal to the side of square A B C D, = 2. Therefore, $2 \times 3{\cdot}125 = 6{\cdot}25$, will be the circumference of circle X. But, the side of square N O P B is, by construction, equal to $\frac{25}{32}$ parts of the side of square A B C D, or diameter of circle X, = 1·5625; and $1{\cdot}5625 \times 4 = 6{\cdot}25$, will be the value of the perimeter of the square N O P B; and is equal to the circumference of circle X.

It may be demonstrated on a false value of the area of a circle, that the area of circle X, is a mean proportional between the area of the circumscribed square A B C D, and the area of a square of which the perimeter is equal to the circumference of the circle X.

For example: Let the area of a circle of which the diameter is unity, be represented by the arithmetical symbols ·7854, and let the perimeter of square N O P B, be equal to the circumference of circle X.

Then, as 1 : ·7854 : : 4 the area of square A B C D, to 3·1416 the area of circle X. As 1 : ·7854 : : 3·1416 the area of circle X, to 2·46741264 the area of square NOPB. And $\sqrt{2 \cdot 46741264} = 1 \cdot 5708$, will be the value of the side of square N O P B. Then, $1 \cdot 5708 \times 4 = 6 \cdot 2832$, will be the value of the perimeter of the square N O P B, and is equal to the circumference of circle X. For, the diameter of circle X is equal to the side of the square $ABCD = 2$. Therefore, $2 \times 3 \cdot 1416 = 6 \cdot 2832$, will be the circumference of circle X, and is equal to the perimeter of the square N O P B. Then, the area of square A B C D is 4; and the area of square N O P B is 2·46741264; and $\sqrt{4 \times 2 \cdot 46741264}, = \sqrt{9 \cdot 86965056}, = 3 \cdot 1416$, is the mean proportional between the area of square A B C D, and the area of square N O P B; and is equal to the area of circle X. The analogy between true and false data, ends with this example.

The area of square E F G H, is equal to half the area of square ABCD, $= 2$. Therefore, $2 \times \cdot 78125 = 1 \cdot 5625$; or, $\frac{25}{32}(2), = \frac{25}{32} \times 2, = \cdot 78125 \times 2, = 1 \cdot 5625$, will be the area of circle Y. But, the side R B, of the square RSTB, is, by construction, equal to four-fifths of the side of square N O P B. Therefore, $\frac{4}{5}(1 \cdot 5625), = \frac{4}{5} \times 1 \cdot 5625, = \cdot 8 \times 1 \cdot 5625, = 1 \cdot 25$, will be the value of the side R B, of the square R S T B; and $1 \cdot 25^2 = 1 \cdot 5625$, will be the area of square R S T B, and is equal to the area of circle Y.

The area of square E F G H is 2. Therefore, the value of the side of the square E F G H, and diameter of circle Y must be $\sqrt{2}$. The side V B, of the square VWZB,

is, by construction, equal to $\frac{25}{32}$ parts of the side of square E F G H, or diameter of circle Y, $= \frac{25}{32}\ (\sqrt{2})$. Therefore, $\frac{25}{32}\ (\sqrt{2}), = \sqrt{\frac{625}{1024} \times 2}, = \sqrt{\cdot 6103515625 \times 2},$ $= \sqrt{1\cdot 220703125}$, will be the value of the side V B, of the square VW Z B; and $\sqrt{1\cdot 220703125}^{\,2}, = 1\cdot 220703125$, will be the area of square V W Z B. But, $\frac{25}{32}$ parts of the area of circle Y, $= \frac{25}{32}\ (1\cdot 5625), = \frac{25}{32} \times 1\cdot 5625, = \cdot 78125 \times 1\cdot 5625, = 1\cdot 220703125$, and is equal to the area of square V W Z B; and the area of circle Y is the mean proportional between the area of square EFGH, and the area of square V W Z B. For, $\sqrt{2 \times 1\cdot 220703125}, = \sqrt{2\cdot 44140625},$ $= 1\cdot 5625$, is the area of circle Y; and is also the mean proportional between the areas of the two squares EFGH and V W Z B.

Again, the diameter of circle Y is equal to the side of square EFGH $= \sqrt{2}$. Therefore, $\sqrt{2} \times 3\cdot 125, = \sqrt{2} \times \frac{25}{8},$ $= \sqrt{2 \times \frac{625}{64}}, = \sqrt{2 \times 9\cdot 765625} = \sqrt{19\cdot 53125}$, will be the value of the circumference of circle Y. But, the side of square V W Z B, is, by construction, equal to $\frac{25}{32}$ parts of the diameter of circle Y, $= \sqrt{1\cdot 220703125}$; and

$$4\ (\sqrt{1\cdot 220703125}), = \sqrt{4^2 \times 1\cdot 220703125}, =$$

$\sqrt{16 \times 1\cdot 220703125}, = \sqrt{19\cdot 53125}$, will be the value of the perimeter of the square V W Z B, and is equal to the circumference of circle Y.

The area of square I K L M, inscribed in circle Y, is equal to half the area of square E F G H, and one-fourth of the area of square A B C D, $= 1$. And among the many relations existing between the figures of which this diagram is composed, the following remarkable ones may be noticed.

The area of square N O P B, is to the area of square R S T B, as the area of circle Y, to the area of the inscribed square I K L M. For, as $2\cdot 44140625 : 1\cdot 5625 ::$

1·5625 : 1. Again, the area of square E F G H, is to the area of the inscribed circle Y, as the area of square R S T B, to the area of square V W Z B. For, as 2 : 1·5625 :: 1·5625 : 1·220703125. Again, the area of circle X, is equal to three and one-eighth times the area of square I K L M.

The most perfect harmony reigns between all the geometrical figures of which the diagram is composed, and the diagram might be extended, outwards or inwards, *ad infinitum.*

I must now direct attention to the diagram No. 10, (see Fig. XIX.) and the following construction of it:

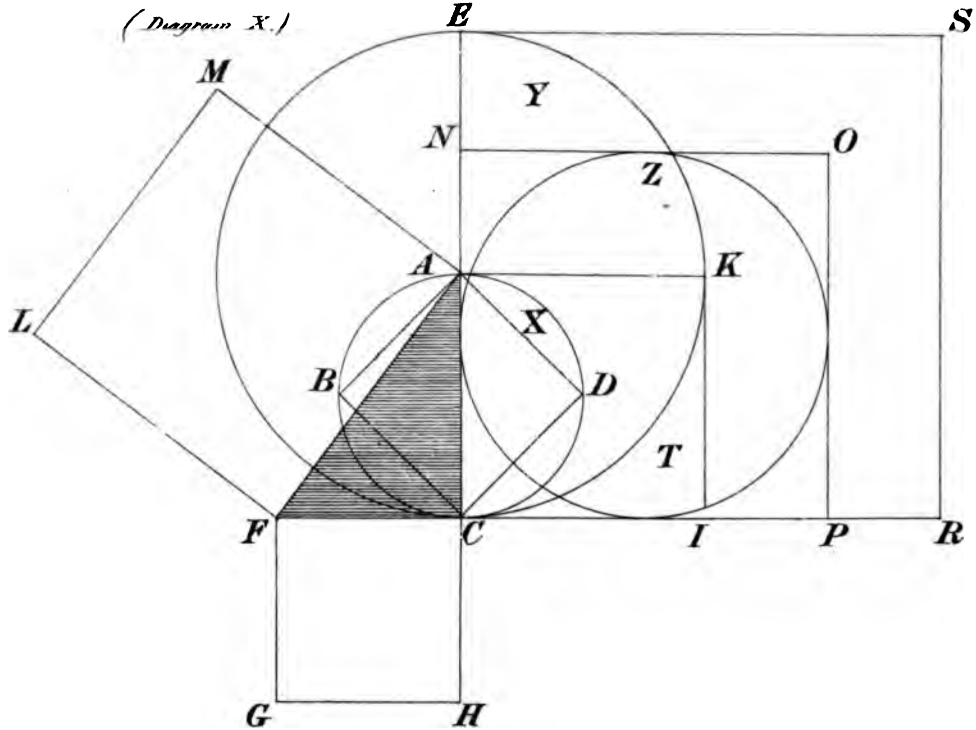
On A B, a given straight line, describe the square A B C D. Draw A C, the diagonal of the square. With A C as diameter, describe the circle X. Produce C A to E, making C E equal to twice C A. From the point A as centre, and with A C or A E as radius, describe the circle Y. On E C, describe the square E C R S. With A C the diameter of circle X, and diagonal of square A B C D, as perpendicular, describe the right-angled triangle A C F, making C F equal to three-fourths of A C. On the sides of the triangle A C F, describe the squares F G H C, A C I K, A F L M. From E C the diameter of circle Y, and a side of the square E C R S, cut off a part N C, making N C equal to three-fourths of E C, and on N C describe the square N O P C. In the square N O P C inscribe the circle Z.

A B, the generating line of the diagram, may be any given number, say 1.

Then, the value of A C, the diagonal of the square A B C D, and diameter of the circle X, will be $\sqrt{2}$; and E C the diameter of circle Y, and a side of the square E C R S, is, by construction, equal to twice the diameter

FIGURE XIX.

(Diagram X.)



of circle X, $= 2\ (\sqrt{2}), = \sqrt{2^2 \times 2}, = \sqrt{4 \times 2}, = \sqrt{8}.$ Then, $\overline{\sqrt{8}}^{2} = 8$, will be the area of square E C R S, described on the diameter of circle Y; and $8 \times {\cdot}78125 = 6{\cdot}25$; or, $\frac{25}{32}(8), = \frac{25}{32} \times 8, = {\cdot}78125 \times 8, = 6{\cdot}25$, will be the area of circle Y. Then, $\overline{\sqrt{2}}^{2} = 2$, will be the area of square A C I K, described on the diameter of circle X; and $2 \times {\cdot}78125, = 1{\cdot}5625$; or, $\frac{25}{32}(2), = \frac{25}{32} \times 2, = {\cdot}78125 \times 2, = 1{\cdot}5625$, will be the area of circle X; and is equal to one-fourth part of the area of circle Y.

The side A C of the triangle A C F, is the diameter of the circle X, the value of which is $\sqrt{2}$; and the side C F of the triangle A C F, is, by construction, equal to three-fourths of A C. Therefore, $\frac{3}{4}(\sqrt{2}), = \sqrt{\frac{9}{16} \times 2}, = \sqrt{{\cdot}5625 \times 2}, = \sqrt{1{\cdot}125}$, will be the value of C F; and $C F^2 + A C^2 = \overline{\sqrt{1{\cdot}125}}^{2} + \overline{\sqrt{2}}^{2}, = 1{\cdot}125 + 2, = 3{\cdot}125$; and $\sqrt{3{\cdot}125}$, will be the value of A F, the hypotheneuse or third side of the triangle A C F.

The value of the side C F, of the triangle A C F, is $\sqrt{1{\cdot}125}$. Therefore, $\overline{\sqrt{1{\cdot}125}}^{2} = 1{\cdot}125$, will be the area of square F G H C. The value of the side A C, of the triangle A C F, is $\sqrt{2}$. Therefore, $\overline{\sqrt{2}}^{2} = 2$, will be the area of square A C I K. The value of the side A F, of the triangle A C F, is $\sqrt{3{\cdot}125}$. Therefore, $\overline{\sqrt{3{\cdot}125}}^{2} = 3{\cdot}125$, will be the area of square A F L M. But, the area of square F G H C, plus the area of square A C I K, plus the are of square A F L M, $= 1{\cdot}125 + 2 + 3{\cdot}125 = 6{\cdot}25$; and the areas of the three squares F G H C, A C I K, A F L M, are together equal to the area of circle Y.

The side N C, of the square N O P C, is, by construction, equal to three-fourths of E C, the diameter of circle Y, and a side of the square E C R S, $= \frac{3}{4}(\sqrt{8})$. Therefore, $\frac{3}{4}(\sqrt{8}) = \sqrt{\frac{9}{16} \times 8} = \sqrt{{\cdot}5625 \times 8}, = \sqrt{4{\cdot}5}$, will be

the value of N C. Then, $\sqrt{4.5}^2 = 4.5$, will be the area of square NOPC; and $4.5 \times .78125, = 3.515625$; or, $\frac{25}{32}(4.5)$, $= \frac{25}{32} \times 4.5, = .78125 \times 4.5, = 3.515625$, will be the area of circle Z; and is equal to three and one-eighth times the area of square FGHC. For, the area of square FGHC is 1.125; and $1.125 \times 3.125 = 3.515625$, is the area of circle Z.

The given value of A B, the side of square A B C D, and the generating line of the diagram, is 1; and the area of square A B C D, must necessarily be 1, the same arithmetical symbol that represents the value of the side of the square. But, the area of square A F L M is 3.125, and is therefore equal to three and one-eighth times the area of square A B C D.

The area of square A C I K, described on the diameter of circle X, is 2. But, the area of circle Y is 6.25, and is therefore equal to three and one-eighth times the area of square A C I K, described on its radius.*

* This fact may be demonstrated in the following way:

The value of the side F C, of the triangle A C F, is $\sqrt{1.125}$; and $2(\sqrt{1.125}) = \sqrt{4.5}$. Let, $\sqrt{4.5}$ be the diameter of a circle. Then, $\sqrt{4.5}^2 = 4.5$, will be the area of a square described on its diameter; and $4.5 \times .78125 = 3.515625$, will be the area of the circle; and is equal to three and one-eighth times the area of square F G H C, described on F C.

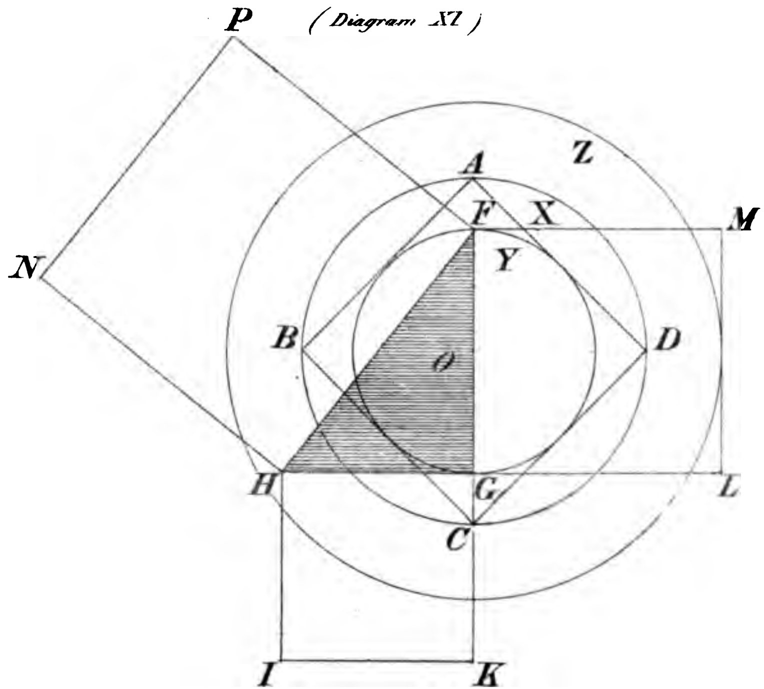
The value of the side A C, of the triangle A C F, is $\sqrt{2}$; and $2(\sqrt{2}) = \sqrt{8}$. Let $\sqrt{8}$ be the diameter of a circle. Then $\sqrt{8}^2 = 8$, will be the area of a square described on its diameter; and $8 \times .78125 = 6.25$, will be the area of the circle; and is equal to three and one-eighth times the area of square A C I K, described on A C.

The value of the side A F, of the triangle A C F, is $\sqrt{3.125}$; and $2(\sqrt{3.125}) = \sqrt{12.5}$. Let $\sqrt{12.5}$ be the diameter of a circle. Then, $\sqrt{12.5}^2 = 12.5$, will be the area of a square described on its diameter; and $12.5 \times .78125 = 9.765625$, will be the area of the circle; and is equal to three and one-eighth times the area of square A F L M, described on A F.

Hence:—The circumference of every circle, is to its diameter; as the

FIGURE XX

(*Diagram XI*)



Again, the area of square A C I K is 2. And 2 × ·78125, = 1·5625; or, $\frac{25}{32}$ (2), = $\frac{25}{32}$ × 2, = ·78125 × 2, = 1·5625, will be the area of an inscribed circle. But, the trapezium A C T K A, inscribed in the square A C I K, is a quadrant of the circle Y, and the area of the trapezium A C T K A is therefore equal to one-fourth part of the area of circle Y. = $\frac{1}{4}$ (6·25), = $\frac{1}{4}$ × 6·25, = ·25 × 6·25, = 1·5625. Hence, the area of the quadrant A C T K A, inscribed in the square A C I K, is equal to the area of a circle inscribed in the same square.

The following is the construction of diagram No. 11, (see Fig. XX.) to which I now beg to direct attention.

On A B, a given straight line, describe the square A B C D. About the square A B C D circumscribe the circle X, and within it inscribe the circle Y. Draw F G, the diameter of circle Y. With the point O as centre, and F G as radius, describe the circle Z. With F G, the diameter of circle Y, as perpendicular, describe the right-angled triangle F G H, making G H equal to three-fourths of F G. On the sides of the triangle F G H, describe the squares G H I K, F G L M, F H N P.

A B, the generating line of the diagram, may be any given number, say 4.

Then, 4^2 = 16, will be the area of square A B C D. $\frac{25}{16}$ (16), = $\frac{25}{16}$ × 16, = 1·5625 × 16, = 25, will be the area of the circumscribed circle X. $\frac{25}{32}$ (16), = $\frac{25}{32}$ × 16, = ·78125 × 16, = 12·5, will be the area of the inscribed circle Y, and is equal to half the area of circle X. The dia-

area of the circle, to the area of a square described on its radius; and may be demonstrated to be so, on any hypothetical data.

For example: Let the circumference of a circle of which the diameter is unity, be 3·1416. Then, the area of the circle will be ·7854. And, as 3·1416 : 1 : : ·7854 : ·25; the exact value of the area of a square, described on a radius of the circle.

meter of circle Y, is equal to the side of the circumscribed square A B C D, $= 4$, and the diameter of circle Z is, by construction, equal to twice the diameter of circle Y, $= 4 \times 2 = 8$. Then, $8^2 = 64$, will be the area of a square described on the diameter of circle Z; and $64 \times \cdot 78125 = 50$; or, $\frac{25}{32}$ (64), $\frac{25}{32} \times 64, = \cdot 78125 \times 64, = 50$, will be the area of circle Z; and is equal to twice the area of circle X, and four times the area of circle Y.

The side F G, of the triangle F G H, is the diameter of circle Y, the value of which is 4; and the side G H, of the triangle F G H is, by construction, equal to three-fourths of the side F G. Therefore, $\frac{3}{4}$ (4), $= \frac{3}{4} \times 4, = \cdot 75 \times 4, = 3$, will be the value of G H; and $\sqrt{GH^2 + FG^2}$, $= \sqrt{3^2 + 4^2}, = \sqrt{9 + 16}, = \sqrt{25}, = 5$; will be the value of F H, the hypotheneuse or third side of the triangle F G H. Then, the value of the side F H, of the triangle F G H, is 5. Therefore, $5^2 = 25$, will be the area of the square F H N P, described on F H, and is equal to the area of circle X.

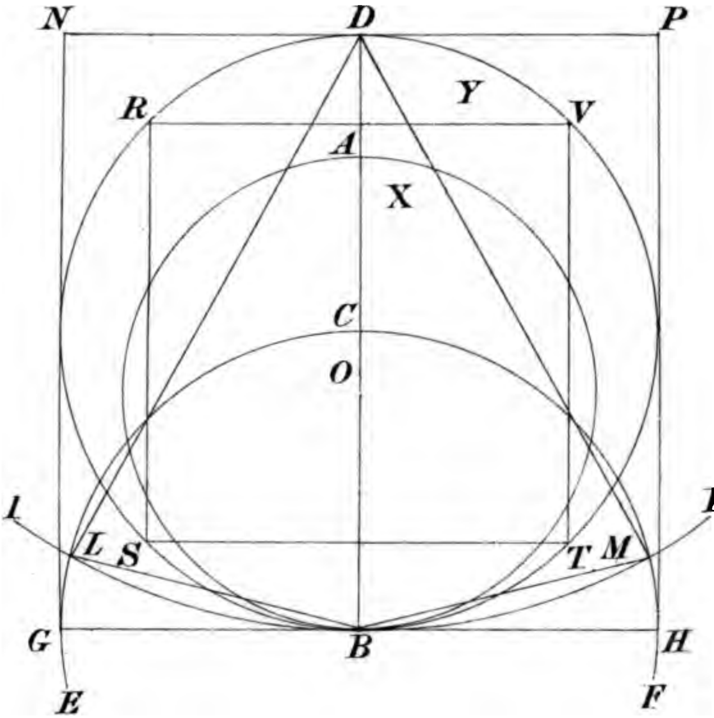
The value of the side F G, of the triangle F G H, is 4. Therefore, $4^2 = 16$, will be the area of square F G L M. The value of the side G H, of the triangle F G H, is 3. Therefore, $3^2 = 9$, will be the area of the square G H I K. But, the arithmetical mean between the area of square F G L M, and the area of square G H I K, is equal to $\frac{1}{2}$ $(16 + 9), = 12\cdot 5$, and is equal to the area of circle Y. The areas of the three squares G H I K, F G L M, F H N P, are together equal to the area of circle Z, as demonstrated in the illustrations of diagram No. 10, (see Fig. XIX.)

Again, the area of circle Y, may be the given quantity, say 288.

Then $\frac{32}{25}$ (288), $\frac{32}{25} \times 288, = 1\cdot 28 \times 288, = 368\cdot 64$, will be the area of the circumscribed square A B C D. $\frac{25}{16}$

FIGURE XXI

(Diagram XII.)



$(368{\cdot}64), = \frac{25}{16} \times 368{\cdot}64, = 1{\cdot}5625 \times 368{\cdot}64, = 576$, will be the area of the circumscribed circle X; and is equal to twice the area of circle Y. The diameter of circle Y, is equal to the side of the circumscribed square A B C D, $= \sqrt{368{\cdot}64} = 19{\cdot}2$; and the diameter of circle Z, is, by construction, equal to twice the diameter of circle Y, $= 19{\cdot}2 \times 2 = 38{\cdot}4$. Then, $38{\cdot}4^2 = 1474{\cdot}56$, will be the area of a square described on the diameter of circle Z; and $\frac{25}{32}(1474{\cdot}56), = \frac{25}{32} \times 1474{\cdot}56, = {\cdot}78125 \times 1474{\cdot}56, = 1152$, will be the area of circle Z; and is equal to twice the area of circle X, and four times the area of circle Y.

The side F G, of the triangle F G H, is the diameter of circle Y, the value of which is 19·2; and the side G H, of the triangle F G H, is, by construction, equal to three-fourths of the side F G. Therefore, $\frac{3}{4}(19{\cdot}2), = \frac{3}{4} \times 19{\cdot}2, = {\cdot}75 \times 19{\cdot}2, = 14{\cdot}4$, will be the value of G H; and $\sqrt{GH^2 + FG^2}, = \sqrt{14{\cdot}4^2 + 19{\cdot}2^2}, = \sqrt{207{\cdot}36 + 368{\cdot}64}, = \sqrt{576}, = 24$; will be the value of F H, the hypotheneuse or third side of the triangle F G H. But, $FH^2 = 24^2, = 576$, will be the area of square F H N P, and is equal to the area of circle X.

The value of any other figure in the diagram may be the given quantity, and if the calculations be worked out, will be found to produce similar results.

I shall now direct attention to a diagram, which introduces to notice certain commensurable relations, between a circle and an isosceles triangle, a geometrical figure to which I have not hitherto made any reference.

Permit me to refer you to the diagram No. 12, (see Fig. XXI.) and the following construction of it.

From the point O as centre, and with any radius, describe the circle X; draw the diameter A B. From A B cut off a part C B, making C B equal to five-eighths

of A B. Produce B A to D, making B D equal to five-fourths of A B. With the point B as centre, and C B as radius, describe the curved line E C F. From the point B, draw B G, B H, at right angles to A B, meeting the curved line E C F, at the points G and H. With the point D as centre, and D B as radius, describe the curved line I B K, cutting the curved line E C F, at the points L and M. Draw D L, L B, D M, M B, describing the isosceles triangles D L B, D M B. From the points G and H, draw G N, H P, parallel and equal to D B. Join N, P, describing the square N G H P. In the square N G H P, inscribe the circle Y; and in the circle Y, inscribe the square R S T V.

A B the diameter of circle X, the generating figure of the diagram, may be any given number, say 8. Then, $8 \times 3{\cdot}125 = 25$, will be the circumference of circle X; $8^2 = 64$, = will be the area of a square described on the diameter of circle X; and $64 \times {\cdot}78125 = 50$, will be the area of circle X. Then, because D L, D B, D M, are all radii of a circle, of which the curved line I B K is a segment, they are all equal; and, because L B, C B, M B, are all radii of a circle, of which the curved line E C F is a segment, they are also all equal. But, D B is, by construction, equal to five-fourths of A B, $= \frac{5}{4}(8), = \frac{5}{4} \times 8, = 1{\cdot}25 \times 8, = 10$. Therefore, the value of D L, and D M, is also 10. C B is, by construction, equal to five-eighths of A B, $= \frac{5}{8}(8), = \frac{5}{8} \times 8, = {\cdot}625 \times 8, = 5$. Therefore, the value of L B and M B is also 5. Then, the value of the sides D L, D B, of the isosceles triangle D L B is 10; and the value of L B is 5. Therefore, D L + D B + L B, $= 10 + 10 + 5, = 25$, is the value of the perimeter of the isosceles triangle D L B; and is equal to the circumference of circle X. The isosceles triangle D M B,

is equal to the isosceles triangle D L B. Therefore, the perimeter of the triangle D M B, is also equal to the circumference of circle X.

Again, because G B, B H, are radii of the same circle as L B, C B, M B, they are all equal. But, the value of L B, C B, M B, is 5. Therefore, the value of G B, B H, is also 5. G N and H P, sides of the square N G H P, are, by construction, equal to D B, and the value of D B is 10. Therefore, the value of G N and H P, is also 10. But, G B and B H, are, by construction, at right angles to D B; and G N and H P, are, by construction, parallel to D B. Therefore, D B G N, and D B H P, are rectangles, of which the value of the longer sides is 10; and the value of the shorter sides is 5; and $10 \times 5 = 50$, will be the area of each of the rectangles D B G N, and D B H P, and is equal to the area of the generating circle X.

Again, the rectangles D B G N, and D B H P, together make up the square N G H P, of which the value of the side must be 10. Therefore, $10^2 = 100$, will be the area of square N G H P; and is equal to twice the area of the generating circle X. Then, $100 \times \cdot78125, = 78\cdot125$; or, $\frac{25}{32}(100), = \frac{25}{32} \times 100, = \cdot78125 \times 100, = 78\cdot125$, will be the area of the inscribed circle Y; and $\frac{16}{25}(78\cdot125), \frac{16}{25} \times 78\cdot125, = \cdot64 \times 78\cdot125, = 50$, will be the area of the square R S T V, inscribed in circle Y; and is equal to half the area of the circumscribed square N G H P, and equal to the area of the generating circle X.

Again, the value of C B, the radius of circle Y, is equal to the value of L B and M B, = 5. But, C B, the radius of circle Y, is equal to the side of a regular hexagon inscribed in the circle. Therefore, L B and M B, the base of the isosceles triangles D L B, and D M B, are also equal to the side of a regular hexagon inscribed in the circle Y.

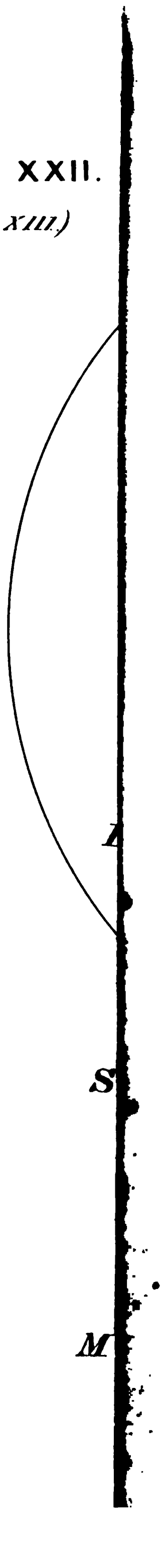
Again, the quadrilateral D L B M D, is described by the lines D L, L B, D M, M B, which represent the base, and one side of the isosceles triangles D L B, and D M B, and of which the value of the sides D L, and D M, is equal to twice the value of the sides L B, and M B. Now, it is conceivable that the perimeter of the quadrilateral D L B M D, might consist of a thin elastic line, fixed at the points D and B, and then, if the line were pressed into the form of a six-sided figure, within the circle Y, so as to touch the circumference of the circle at six equi-distant points, it would then describe an inscribed regular hexagon, within the circle Y.

Again, the diameter of circle Y, is equal to the side of the circumscribed square N G H P; and D B, a diameter of circle Y, divides the circle into two equal parts, each of which is therefore equal in area to the area of the semi-circle G C H; for, C B, the radius of the semi-circle G C H, is also the radius of circle Y, and the area of the semi-circle G C H, is therefore equal to half the area of circle Y, $= \frac{1}{2}(78{\cdot}125), = 39{\cdot}0625$. Then, the diameter of the generating circle X is 8. Therefore, $8^2 = 64$, will be the area of a square described on the diameter of circle X; and the area of circle X, is the mean proportional between the area of a square described on its diameter, and the area of the semi-circle G C H, or either of the semi-circles into which the circle Y is divided, by D B the diameter. For, $\sqrt{39{\cdot}0625 \times 64}, = \sqrt{2500}, = 50$, and is equal to the area of the generating circle X.

Hence, by means of this diagram, the following facts are demonstrated: If the sides of an isosceles triangle, be equal to twice the base, the perimeter of the triangle, is equal to the circumference of a circle, of which the diameter is equal to four-fifths of the side of the triangle, or eight-fifths of its base.

FIGURE XXII.

(Diagram XIII.)



The area of a rectangle, of which the value of the longer sides, is equal to twice the value of the shorter sides; is equal in area to the area of a circle, of which the diameter is equal to four-fifths of the longer sides, or eight-fifths of the shorter sides of the rectangle.

The ingenious inquirer after truth may extend this diagram, and pursue the examination of the question in this particular direction, with considerable advantage.

I must now beg you to refer to the diagram No. 13, (see Fig. XXII.) and the following construction of it:

With the point A as centre, and with any radius, describe the circle X. Draw A B, a radius of the circle. With the point B as centre, and B C, equal to half of A B, as radius, describe the circle Y. The circles cut each other at the point D. Draw A D, B D, describing the isosceles triangle A B D. Draw A E, bisecting the angle A, and also bisecting B D, the base of the triangle A B D, at F. From the point E, draw E G, E H, at right-angles to A E, making E G, E H, equal to F B, F D. From the points G and H, draw G I, H K, parallel and equal to A E. Join I, K, describing the rectangle I G H K. Draw B E, D E, describing the inscribed quadrilateral A B E D A. About the circle Y, circumscribe the square L M N P; and in the circle Y, inscribe the square R S T D. With the point B as centre, and B V, equal to four-fifths of B R or B C, as radius, describe the circle Z.

A B, the radius of circle X, may be any given number, say 10. Then, B C, the radius of circle Y, is, by construction, equal to half of A B, the radius of circle X = 5. And B V, the radius of circle Z, is, by constructition, equal to four-fifths of B C, the radius of circle Y, = 4. Therefore, 4 × 2 = 8, will be the diameter of circle

Z. Then, $8 \times 3{\cdot}125 = 25$, will be the circumference of circle Z; $8^2 = 64$, will be the area of a square described on the diameter of circle Z; and $64 \times {\cdot}78125 = 50$, will be the area of circle Z.

Then, because A B, A E, A D, are radii of the circle X, they are all equal $= 10$. And, because B C, B D, are radii of the circle Y, they are also equal, and are, by construction, equal to half of A B, the radius of circle X $= 5$. And A B D is an isosceles triangle, of which the value of each of the sides is equal to twice the value of the base. Therefore, A B + A D + B D, $= 10 + 10 + 5 = 25$, is the value of the perimeter of the isosceles triangle A B D, and is equal to the circumference of circle Z.

Again, E G and E H, are, by construction, equal to F B and F D; and F B + F D, is equal to B D, the base of the isosceles triangle A B D. Therefore, G H, equal to E G + E H, is also equal to B D, the base of the isosceles triangle A B D $= 5$. G I and H K, are, by construction, equal and parallel to A E. But, A E is a radius of the circle X, and equal to A B, and A D, the sides of the isosceles triangle A B D. Therefore, G I, and H K, are also equal to A B, and A D, the sides of the isosceles triangle A B D $= 10$; and I G H K is a rectangle, of which the value of the longer sides, G I and H K, is 10; and the value of the shorter sides, G H and I K, is 5; and $10 \times 5 = 50$, is the area of the rectangle I G H K; and is equal to the area of circle Z.

Again, the radius of circle Y, is, by construction, equal to the half of A B, the radius of circle X, $= 5$, and $5 \times 2 = 10$, will be the value of the diameter of circle Y, and the side of square L M N P, circumscribed about the circle Y. Then, $10^2 = 100$, will be the area of square L M N P. $\frac{25}{32}(100), = \frac{25}{32} \times 100, = {\cdot}78125 \times 100, = 78{\cdot}125$,

will be the area of circle Y. $\frac{16}{25}$ (78·125), $= \frac{16}{25} \times 78·125, =$ $·64 \times 78·125, = 50$, will be the area of the square RSTD, inscribed in circle Y, and is equal to the area of circle Z.

Again, it is evident that the area of the rectangle I G H K, is equal to twice the area of the inscribed quadrilateral A B E D A. Now, suppose a square to be inscribed in circle Z, and a circle to be inscribed in such square, the area of the circle will be equal to the area of the quadrilateral A B E D A. Or, suppose a circle to be inscribed in the square R S T D, and a square to be inscribed in such circle, the area of the square will also be equal to the area of the quadrilateral A B E D A.

Again, the value of A B, the radius of circle X, is 10; and $10 \times 2 = 20$, will be the value of the diameter of circle X. Then, $20^2 = 400$, will be the area of a square circumscribed about the circle X, and is equal to eight times the area of the rectangle I G H K, or the area of the square R S T D. The area of a square inscribed in the circle X, will be 200, and is equal to four times the area of the rectangle I G H K, or the area of the square R S T D. But, I beg to direct your especial attention to the following very remarkable differences. Eight rectangles, each being in all respects equal to the rectangle I G H K, may be united, so as to describe a square, equal to a square circumscribed about the circle X; but four such rectangles, although equal in area to the area of a square inscribed in circle X, cannot be united, so as to describe such square. Four squares, each being in all respects equal to the square R S T D, may be united, so as to describe a square, equal to a square inscribed in the circle X; but eight such squares, although equal in area to the area of a square circumscribed about the circle X, cannot be united, so as to describe such square.

Again, the area of a square circumscribed about the circle X, will be 400. The area of a square inscribed in circle X, will be 200. The area of square L M N P, is equal to the area of a circle described on the radius of circle X = 100. And the area of a square inscribed in circle X, is the mean proportional between the area of a circumscribed square, and the area of a square described on a radius of the circle. For $\sqrt{400 \times 100} = \sqrt{40000} =$ 200, and is equal to the area of a square inscribed in circle X.

You may perhaps tell me that all these latter facts are known to, and are not disputed by, the Mathematician. Be it so. Then, I put the question:—Can the Mathematician be so deluded, as not to be able to perceive, that such facts could not possibly exist, if there were no definable relation between the diameter and circumference of a circle?

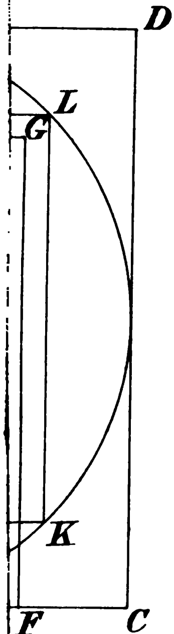
The demonstrations, by means of the diagram No. 14, (see Fig. XXIII.) are to me extremely interesting, and to them I would now request especial attention.

The circles *a*, *b*, *c*, *d*, *e*, represent a succession of circles, each containing twice the area of the one that precedes it.

Let the circumference of circle *a*, be 4.

Then, $4 \div 3{\cdot}125 = 1{\cdot}28$, will be the diameter of circle *a;* and $\frac{1}{2}(4) \times \frac{1}{2}(1{\cdot}28), = 2 \times {\cdot}64, = 1{\cdot}28$, will be the area of circle *a;* and the diameter and area of circle *a*, are represented by the same arithmetical symbols. I have directed attention to this fact in my pamphlet, and refer the reader to what I have there demonstrated.

It is evident that the area of circle *e*, is equal to sixteen times the area of circle *a*, $= 1{\cdot}28 \times 16 = 20{\cdot}48$. Then, $\frac{32}{25}(20{\cdot}48), \frac{32}{25} \times 20{\cdot}48, = 1{\cdot}28 \times 20{\cdot}48, = 26{\cdot}2144$;



D
L
G
K
F
C

or, as $\cdot 78125 : 1 :: 20\cdot 48 : 26\cdot 2144$; and $26\cdot 2144$, will be the area of the square P R S B, circumscribed about the circle *e*. And, $\sqrt{26\cdot 2144} = 5\cdot 12$, will be the value of the side of square P R S B, and diameter of circle *e*.

Produce B P, a side of the square P R S B to A, making A B equal to $\frac{25}{16}$ parts of P B, $= \frac{25}{16}(5\cdot 12), = \frac{25}{16} \times 5\cdot 12, = 1\cdot 5625 \times 5\cdot 12, = 8$; and 8 will be the value of A B, the side of the square A B C D.

Then, $8^2 = 64$, will be the area of square A B C D. $\frac{25}{32}(8), = \frac{25}{32} \times 8, = \cdot 78125 \times 8, = 50$, will be the area of circle X; $\frac{16}{25}(50), = \frac{16}{25} \times 50, = \cdot 64 \times 50, = 32$, will be the area of the inscribed square H I K L, and is equal to half the area of square A B C D.

The area of circle X is 50. The area of circle *e* is $20\cdot 48$. And $\sqrt{50 \times 20\cdot 48}, = \sqrt{1024}, = 32$, is the area of the square H I K L, inscribed in the circle X; and the area of square H I K L is the mean proportional between the area of circle X, and the area of circle *e*.

Hence, the area of every square, is a mean proportional, between the area of a circle circumscribed about it, and the area of a circle of which the diameter is equal to $\frac{16}{25}$ parts of the diameter of the circumscribed circle.*

Again, the area of every circle, is a mean proportional, between the area of a square circumscribed about it, or, described on its diameter, and the area of a square of which the perimeter is equal to the circumference of the circle; and this may be demonstrated, either on a true or false value of the circumference and area of a circle.

* At this point I should have directed attention to the fact, that the area of square A B C D, is equal to three and one-eighth times the area of circle *e*. Hence, the area of every square is equal to three and one-eighth times the area of a circle, of which the diameter is equal to $\frac{16}{25}$ parts of the side of the square.

For example: Let the perimeter of the square EBFG, be equal to the circumference of circle X, and let the circumference of a circle of which the diameter is unity be 3·1416, and its area ·7854.

The value of the side of square ABCD, and diameter of circle X, is 8.

Then, $8^2 = 64$, will be the area of square A B C D. $8 \times \cdot 31416 = 25\cdot 1328$, will be the circumference of circle X. As 1 : ·7854 : : 64 the area of square A B C D, to 50·2656; or, $64 \times \cdot 7854$, also $= 50\cdot 2656$, will be the area of circle X. As 1 : ·7854 : : 50·2656, the area of circle X, to 39·47860224; or, $50\cdot 2656 \times 64$, also $= 39\cdot 47860224$, will be the area of the square E F G H; $\sqrt{39\cdot 47860224} = 6\cdot 2832$, must be the value of the side of the square EBFG; and $6\cdot 2832 \times 4 = 25\cdot 1328$, must be the value of the perimeter of square E B F G; and is equal to the circumference of circle X.

Then, on this data, the area of square A B C D, is 64. The area of square E B F G, is 39·47860224. And $\sqrt{64 \times 39\cdot 47860224}, = \sqrt{2526\cdot 63054386}, = 50\cdot 2656$; and is equal to the area of circle X. And the area of circle X is the mean proportional between the area of the circumscribed square A B C D, and the area of the square E B F G, of which the perimeter is equal to the circumference of circle X.

Any other hypothetical data may be taken to represent the circumference and area of a circle, of which the diameter is unity, and if the calculations be worked out, they will be found to produce a similar result.

Again, let the side of square A B C D be 9·9. Then, $9\cdot 9^2 = 98\cdot 01$, will be the area of square ABCD; $\frac{25}{32}(98\cdot 01) = \frac{25}{32} \times 98\cdot 01, = \cdot 78125 \times 98\cdot 01, = 76\cdot 5703125$, will be the area of circle X; $\frac{16}{25}(76\cdot 5703125), = \frac{16}{25} \times 76\cdot 5703125,$

$= \cdot 64 \times 76\cdot5703125, = 49\cdot005$, will be the area of the inscribed square H I K L; and is equal to half the area of square A B C D.

Let the side of the square M B N O, be equal to the radius of circle X; it will be equal to half the side of square A B C D, $= \frac{1}{2}(9\cdot9) = 4\cdot95$; and $4\cdot95^2 = 24\cdot5025$, will be the area of the square M N O B.

Then, the area of square A B C D is $98\cdot01$. The area of square M B N O is $24\cdot5025$. And $\sqrt{98\cdot01 \times 24\cdot5025}$, $= \sqrt{2401\cdot490025}, = 49\cdot005$; and is equal to the area of the square H I K L. And the area of square H I K L, is the mean proportional between the area of square A B C D, circumscribed about the circle X, and the area of square M B N O, described on a radius of circle X.

Hence, if a square be inscribed in any circle, the area of such square is a mean proportional between the area of a square circumscribed about the circle, and the area of a square described on a radius of the circle.

In the last example, the comparison has reference to three squares, and it is impracticable to adopt false data. In the first example, the comparison has reference to two circles and a square, and it is impossible to arrive at corret results, except on true data. In the intermediate example, the comparison has reference to two squares and a circle, and the facts may be demonstrated on either true or false data.*

* I have already directed attention, in a note, to one fact which had escaped my notice when writing the letter, and I may now observe that I have omitted to notice some very important facts, of the particular character to which this diagram has a special reference, and to which I would now request attention, as being of the utmost importance in the consideration of this question.

I have stated that the circles *a*, *b*, *c*, *d*, *e*, represent a succession of cir-

The demonstrations by means of diagram No. 15, (see Fig. XXIV.) are of the utmost importance, and merit

cles, each containing twice the area of the one that precedes it. Take any three of these adjacent circles, say the circles c, d, and e, and let the area of the intermediate circle d be 32. Then, the area of circle e will be 64, and the area of circle c will be 16; and $\sqrt{64 \times 16}$, $= \sqrt{1024}$, $= 32$, is the area of the intermediate circle d, and is the mean proportional between the area of circle e and the area of circle c.

The area of circle e is 64. Therefore, $\frac{32}{25}(64)$, $= \frac{32}{25} \times 64$, $= 1{\cdot}28 \times 64$, $= 81{\cdot}92$, will be the area of a square circumscribed about the circle e; and $\sqrt{81{\cdot}92}$ will be the value of the diameter of circle e. Therefore, $3\frac{1}{8}(\sqrt{81{\cdot}92})$, $= \frac{25}{8}(\sqrt{81{\cdot}92})$, $= \sqrt{\frac{625}{64} \times 81{\cdot}92}$, $= \sqrt{9{\cdot}765625 \times 81{\cdot}92}$, $= \sqrt{800}$, will be the circumference of circle e.

The area of square d is 32. Therefore, $\frac{32}{25}(32)$, $= \frac{32}{25} \times 32$, $= 1{\cdot}28 \times 32$, $= 40{\cdot}96$, will be the area of a square circumscribed about the circle d; $\sqrt{40{\cdot}96} = 6{\cdot}4$, will be the diameter of circle d; and $6{\cdot}4 \times 3{\cdot}125 = 20$, will be the circumference of circle d.

The area of circle c is 16. Therefore, $\frac{32}{25}(16)$, $= \frac{32}{25} \times 16$, $= 1{\cdot}28 \times 16$, $= 20{\cdot}48$, will be the area of a square circumscribed about the circle c; and $\sqrt{20{\cdot}48}$, will be the diameter of circle c. Therefore, $3\frac{1}{8}(\sqrt{20{\cdot}48})$, $= \frac{25}{8}(\sqrt{20{\cdot}48})$, $= \sqrt{\frac{625}{64} \times 20{\cdot}48}$, $= \sqrt{9{\cdot}765625 \times 20{\cdot}48}$, $= \sqrt{200}$; will be the circumference of circle c.

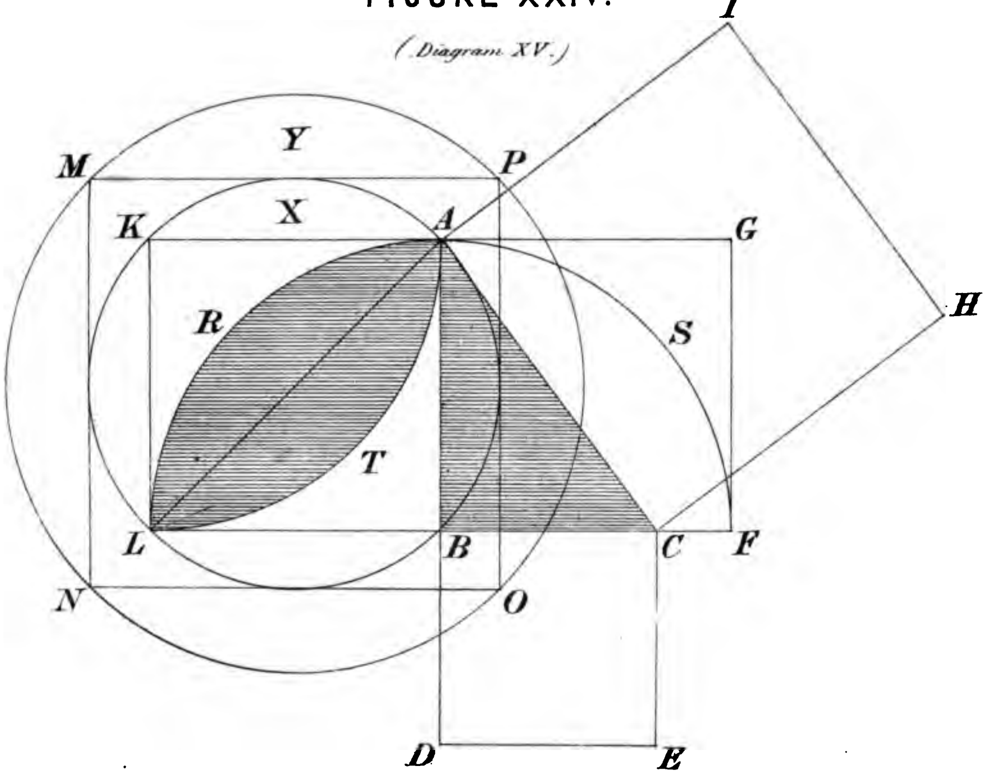
Then, the circumference of circle e is $\sqrt{800}$. The circumference of circle c is $\sqrt{200}$. And $\sqrt{(\sqrt{800} \times \sqrt{200})}$, $= \sqrt{(\sqrt{160000})}$, $= \sqrt{400} = 20$, is the circumference of the intermediate circle d, and is the mean proportional between the circumference of the circle e, and the circumference of the circle c.

Again, the diameter of circle e is $\sqrt{81{\cdot}92}$. The diameter of circle d is $6{\cdot}4$. The diameter of circle c is $\sqrt{20{\cdot}48}$. And $\sqrt{(\sqrt{81{\cdot}92} \times \sqrt{20{\cdot}48})}$ $= \sqrt{(\sqrt{1677{\cdot}7216})} = \sqrt{40{\cdot}96} = 6{\cdot}4$, is the diameter of the intermediate circle d; and is the mean proportional between the diameter of circle e, and the diameter of circle c.

Again, let A, B, C, represent three squares, circumscribed about the circles c, d, e. The area of the intermediate square B, will be a mean proportional between the area of square A, and the area of square C. The perimeter of the intermediate square B, will be a mean proportional between the perimeter of square A, and the perimeter of square C. The side of the intermediate square B, will be a mean proportional between the side of square A, and the side of square C.

FIGURE XXIV.

(Diagram XV.)



a very careful examination. The following is the construction of the diagram :

On the straight line A B, describe the right-angled triangle A B C, making B C, equal to three-fourths of A B. On the sides of the triangle A B C, describe the squares A B F G, B D E C, and A C H I. On A B, describe the square A B L K, and about it circumscribe the circle X. About the circle X, circumscribe the square M N O P; and about the square M N O P, circumscribe the circle Y. With the point B as centre, and A B as radius, describe the semi-circle L R A S F, forming the inscribed quadrants B L R A B, and B A S F B, in the squares A B L K and A B F G. With the point K as centre, and K L or K A as radius, draw the curved line A T L, describing the quadrant K A T L K, and the trapezium A T L R A, in the square A B L K. Draw A L, the diagonal of the square A B L K.

In the first place, I must direct attention to certain peculiar facts, with reference to this diagram.

Let the side A B, of the right-angled triangle A B C, be 4. The side B C, is, by construction, equal to three-fourths of A B, $= \frac{3}{4}$ (4), $= \frac{3}{4} \times 4, = \cdot 75 \times 4, = 3$; and $\sqrt{AB^2 + BC^2}, = \sqrt{4^2 + 3^2}, \sqrt{= 16 + 9}, = \sqrt{25}, = 5$; will be the value of A C, the third side of the triangle A B C ; and this triangle represents the first commensurable right-angled triangle, of which the values of the three sides can be given in integers or whole numbers.

Then, the value of A C is 5, and the value of B C is 3, and the arithmetical mean between these two numbers, equal to $\frac{1}{2}(5 + 3) = 4$, is the given value of A B, the remaining side of the triangle A B C ; and in every right-angled triangle, of which the two sides adjacent to the right angle, are in the ratio of 3 to 4, the value of the longer

of the two sides, adjacent to the right angle, is the arithmetical mean between the values of the other two sides.

Again, Let the value of the side A B, of the right-angled triangle A B C, be 1; and the sides B C, A B, adjacent to the right-angle, in the ratio of 3 to 4. Then, $A B^2 = 1^2 = 1$, will be the area of square A B F G. $\frac{3}{4}(1), = \frac{3}{4} \times 1, = \cdot 75 \times 1, = \cdot 75$, will be the value of the side B C, of the triangle A B C; and $\cdot 75^2 = \cdot 5625$, will be the area of the square B D E C, described on the side B C, of the triangle A B C. And $\sqrt{A B^2 + B C^2}, = \sqrt{1^2 + \cdot 75^2}, = \sqrt{1 + \cdot 5625}, = \sqrt{1 \cdot 5625}, = 1 \cdot 25$, will be the value of the side A C, of the triangle A B C; and $1 \cdot 25^2 = 1 \cdot 5625$, will be the area of the square A C H I, described on A C, the third side of the triangle A B C.

Again, Let the value of the side A B, of the right-angled triangle A B C, be $\sqrt{1}$; and the sides B C, A B, adjacent to the right angle, in the ratio of 3 to 4. Then, $A B^2, = \sqrt{1}^{\,2}, = 1$, will be the area of the square A B F G, described on the side A B, of the triangle A B C; $\frac{3}{4}(\sqrt{1}), = \sqrt{\frac{9}{16} \times 1}, = \sqrt{\cdot 5625 \times 1}, = \sqrt{\cdot 5625}$, will be the value of the side B C; and $B C^2 = \sqrt{\cdot 5625}^{\,2} = 5625$, will be the area of the square B D E C, described on the side B C, of the triangle A B C; and $A B^2 + B C^2, = \sqrt{1}^{\,2} + \sqrt{\cdot 5625}^{\,2}, = 1 + \cdot 5625, = 1 \cdot 5625$; will be the area of the square A C H I, described on A C, the third side of the triangle A B C; and whether we adopt 1 or $\sqrt{1}$, as the value of A B, of the triangle A B C, we arrive at the same values of the areas of the squares described on its sides; but we have worked out this result, by means of different arithmetical symbols.

Then, $\sqrt{1 \cdot 5625}, = 1 \cdot 25$, is the value of the side A C; and $\sqrt{\cdot 5625} = \cdot 75$, is the value of the side B C, in the triangle A B C; and the arithmetical mean between these

two numbers, equal to $\frac{1}{2}$ (1·25 + ·75) = 1, is the given value of A B, the remaining side of the triangle A B C.

The area of square A B F G, and the area of square B D E C, are together equal to the area of square A C H I. Hence, it is evident, that if the areas of any two geometrical figures, of whatever form, be together equal to the area of a third geometrical figure, the areas of the three figures will be severally equal to the areas of squares, described on the sides of a right-angled triangle; and the square root of the areas of the three figures, will give the value of the sides of such triangle; and if the value of the middle side, be the arithmetical mean between the values of the other two sides, the triangle will be one, of which the two sides adjacent to the right angle, are in the ratio of 3 to 4.

In my Pamphlet I have shewn, that from any two given numbers, a commensurable right-angled triangle may be obtained; but from commensurable right-angled triangles, so obtained, in no case will the two sides adjacent to the right angle be in the ratio of 3 to 4, with one exception. (The exception being, when the given numbers are 1 and 2, which produce the first commensurable right-angled triangle, of which the values of the sides can be given in integers or whole numbers.)

In treating of diagram No. 10, (see Fig. XIX.) I have demonstrated, and here assume it to be admitted, that the quadrants A R L B A, and A T L K A, inscribed in the square A K L B, are each equal in area, to the area of a circle inscribed in the same square.

Let A B, the side of square A K L B, be 4, and let the area of a circle of which the diameter is unity, be ·78125.

Then, $4^2 = 16$, will be the area of square A K L B;

and $16 \times \cdot 78125, = 12 \cdot 5$, will be the area of an inscribed circle. The area of the quadrant A R L B A, inscribed in square A K L B, is equal to the area of a circle inscribed in the square $= 12 \cdot 5$. Therefore, $16 - 12 \cdot 5, = 3 \cdot 5$, will be the area of the trapezium A K L R A; being that part of the square A K L B, outside the quadrant A R L B A. And $16 - 12 \cdot 5, = 3 \cdot 5$, will also be the area of the trapezium A B L T A; being that part of the square A K L B, outside the quadrant A T L K A. Then, $3 \cdot 5 + 3 \cdot 5, = 7$, will be the area of the two parts of the square A K L B, outside the trapezium A R L T A. And it is evident, that the area of the square A K L B, minus the area of the two parts of it, outside the trapezium A R L T A, $= 16 - 7, = 9$, must be the area of the trapezium A R L T A; and the area of square A K L B, plus the area of the trapezium A R L T A, $= 16 + 9, = 25$; and is equal to the area of circle X. For, A L the diagonal of the square A K L B, is equal to the diameter of circle X; and the angles of the square A K L B are right angles. Take any two adjacent sides of the square, say A B, A K. Then, $\sqrt{AB^2 + AK^2}, = \sqrt{4^2 + 4^2}, = \sqrt{16 + 16}, = \sqrt{32}$, will be the value of A L, the diagonal of square A K L B, and the diameter of circle X. Then, the diameter of circle X, is equal to the side of square M N O P, $= \sqrt{32}$; and $\sqrt{32}^2 = 32$, will be the area of the square M N O P, circumscribed about the circle X; and $32 \times \cdot 78125, = 25$, will be the area of circle X; and is equal to the area of the trapezium A R L T A, plus the area of the square A K L B, $= 9 + 16 = 25$.

Let A B, the side of square A K L B, be 4, and let the area of a circle of which the diameter is unity, be $\cdot 7854$.

Then, $4^2 = 16$, will be the area of square A K L B; and $16 \times \cdot 7854, = 12 \cdot 5664$, will be the area of an inscribed

circle. The area of the quadrant A R L B A, inscribed in square A K L B, is equal to the area of a circle inscribed in the square, $= 12{\cdot}5664$. Therefore, $16 - 12{\cdot}5664, = 3{\cdot}4336$, will be the area of the trapezium A K L R A; being that part of the square A K L B, outside the quadrant A R L B A. And $16 - 12{\cdot}5664, = 3{\cdot}4336$, will also be the area of the trapezium A B L T A; being that part of the square A K L B, outside the quadrant A T L K A. Then, $3{\cdot}4336 + 3{\cdot}4336, = 6{\cdot}8672$, will be the area of the two parts of the square A K L B, outside the trapezium A R L T A. And the area of the square A K L B, minus the two parts of it, outside the trapezium A R L T A, $= 16 - 6{\cdot}8672, = 9{\cdot}1328$, will be the area of the trapezium A R L T A; and the area of square A K L B, plus the area of the trapezium A R L T A, $= 16 + 9{\cdot}1328, = 25{\cdot}1328$; and is equal to the area of circle X. For, A L the diagonal of square A K L B, is equal to the diameter of circle X; and the angles of the square A K L B are right angles. Take any two adjacent sides of the square, say A B, A K. Then, $\sqrt{AB^2 + AK^2}, = \sqrt{4^2 + 4^2}, = \sqrt{16 + 16}, = \sqrt{32}$, will be the value of A L, the diagonal of square A K L B, and the diameter of circle X. Then, the diameter of circle X, is equal to the side of square M N O P, $= \sqrt{32}$, and $\overline{\sqrt{32}}^2$, $= 32$, will be the area of the circumscribed square M N O P; and $32 \times {\cdot}7854, = 25{\cdot}1328$, will be the area of circle X; and is equal to the area of the trapezium A R L T A, plus the area of the square AKLB $= 9{\cdot}1328 + 16, = 25{\cdot}1328$.

By both examples it is demonstrated, that the area of circle X, is equal to the area of square A K L B, plus the area of the trapezium ARLTA; and on any other hypothetical data, taken to represent the area of a circle of which the diameter is unity, it may be demonstrated,

that the area of the trapezium A R L T A, and the area of the square A K L B, are together equal to the area of the circle X, circumscribed about the square A K L B.

Again, the side A B, of the square A K L B, is a side of the square A B F G, and also a side of the right-angled triangle A B C, the value of which is 4. Therefore, $4^2 = 16$, will be the area of the square A B F G, described on the side A B, of the triangle A B C. The side B C, of the triangle A B C, is, by construction, equal to three-fourths of the side A B, $= \frac{3}{4} (4), = \frac{3}{4} \times 4, = \cdot 75 \times 4, = 3$. Therefore, $3^2 = 9$, will be the area of square B D E C, described on the side B C, of the triangle A B C. And $AB^2 + BC^2, = 4^2 + 3^2, = 16 + 9 = 25$, will be the area of the square A C H I, described on A C, the third side of the triangle A B C; and is equal to the areas of the squares A B F G, and B D E C, on the other two sides of the triangle.

Thus, we have two distinct groups of geometrical figures in the diagram, each consisting of three figures, and of which the areas of two, are together equal to the area of the third; the one group composed of the circle X, the square A K L B, and the inscribed trapezium A R L T A; and the other composed of the squares A C H I, A B F G, and B D E C, described on the sides of the right-angled triangle A B C. And I have shown that these facts may be demonstrated upon either true or false data.

Again, the value of the side A C, in the triangle A B C, is equal to $\sqrt{BC^2 + AB^2}, = \sqrt{3^2 + 4^2}, = \sqrt{9 + 16}, = \sqrt{25}, = 5$; and $5^2 = 25$, will be the area of the square A C H I; and on the true definition of the relations of a circle, is exactly equal to the area of circle X.

Then, the area of circle X is 25; and, as $\cdot 78125 : 1 :: 25 : 32$, the area of the circumscribed square M N O P; or, on the orthodox data, the area of circle X is 25·1328;

and as $\cdot 7854 : 1 :: 25\cdot 1328 : 32$, the area of the circumscribed square M N O P; so that whether the area of the square M N O P, be ascertained by one or other of these data, no question can arise as to 32 being the true value of its area.

It is evident, that the area of the square A K L B, and the areas of the three trapeziums A R L T A, A R L K A, and A T L B A, inscribed in it, are together equal to twice the area of the square A K L B, and are therefore equal to the area of the square M N O P.

Then, on the true theory of the value of a circle, the area of square A K L B, minus the area of the trapezium A R L T B, is equal to $16 - 9, = 7$; and this difference, plus the area of circle X, or the area of square A C H I, in either case equal to $7 + 25 = 32$; and is equal to the area of the square M N O P.

On the orthodox data, the area of square A K L B, minus the area of the trapezium A R L T A, is equal to $16 - 9\cdot 1328, = 6\cdot 8672$. But, this difference, plus the area of square A C H I, $= 6\cdot 8672 + 25, = 31\cdot 8672$; and in this case is less than the area of square M N O P.

Again, I have demonstrated, both on a true and false value of the area of a circle, that the area of square A K L B, and the area of the trapezium A R L T A, are together equal to the area of circle X. It is therefore evident that the area of circle X, minus the area of the trapezium A R L T A, must be equal to the area of the square A K L B.

Then, on the true theory of the value of a circle, the area of circle X, or the area of square A C H I, minus the area of the trapezium A R L T A, in either case equal to $25 - 9, = 16$; and is exactly equal to the admitted value of the area of square A K L B.

On the orthodox data, the area of square A C H I, minus the area of the trapezium A R L T A, = 25 — 9·1328, = 15·8672; and in this case is less than the admitted value of the area of square A K L B.

Again, on the true theory of the value of a circle, the area of square B D E C, or, the area of the trapezium A R L T A, plus the area of the square A K L B, is in either case, equal to 9 + 16 = 25, and is equal to the area of circle X, or the area of square A C H I.

On the orthodox data, the area of the trapezium A R L T A, plus the area of the square A K L B, is equal to 9·1328 + 16 = 25·1328, and in this case is greater than the admitted value of the area of the square A C H I.

Again, the side of the square M N O P, is equal to the diameter of circle X, $= \sqrt{32}$. Take two adjacent sides of the square, say M P, M N. Then, $\sqrt{MP^2 + MN^2}$, $= \sqrt{(\sqrt{32}^2 + \sqrt{32}^2)}$, $= \sqrt{32 + 32}$, $= \sqrt{64}$, $= 8$, will be the value of the diagonal of the square M N O P, and is equal to the diameter of circle Y. Then, $8^2 = 64$, will be the area of a square described on the diameter of circle Y; and 64 × ·78125, = 50, will be the area of circle Y.

On the orthodox data $8^2 = 64$, will be the area of a square described on the diameter of circle Y; and 64 × ·7854 = 50·2656, will be the area of circle Y. On either data the area of circle Y, is equal to twice the area of circle X.

The side A B, of the triangle A B C, the value of which is 4, is common to the two squares A K L B, and A G F B, the two squares are therefore in all respects equal; and $4^2 = 16$, is the value of the area of both.

Then, on the true theory of the value of a circle, the area of square A C H I, plus the area of square A B F G, plus the area of square B D E C, is equal to (25 + 16 + 9) = 50. The area of circle X, plus the area of square

A K L B, plus the area of the trapezium A R L T A, is equal to (25 + 16 + 9) = 50. The area of square A C H I, plus the area of square A K L B, plus the area of the trapezium A R L T A, is equal to (25 + 16 + 9) = 50. The area of circle X, plus the area of square A B F G, plus the area of square B D E C, is equal to (25 + 16 + 9) = 50. The area of square A C H I, plus the area of square A K L B, plus the area of square B D E C, is equal to (25 + 16 + 9) = 50. The area of circle X, plus the area of square A B F G, plus the area of the trapezium A R L T A, is equal to (25 + 16 + 9) = 50. And in all these instances we have examples of distinct sets of three geometrical figures contained in this diagram, the areas of which are together equal to the area of circle Y.

On the orthodox data, the area of circle Y is 50·2656. But, the area of square A C H I, plus the area of square A B F G, or A K L B, plus the area of square B D E C, is equal to (25 + 16 + 9) = 50. The area of square A C H I, plus the area of square A B F G, or A K L B, plus the area of the trapezium A R L T A, is equal to (25 + 16 + 9·1328) = 50·1328. In these instances, the values differ from each other, and in both cases are less than the value of the area of circle Y, as ascertained on the orthodox data.

I shall now reverse the operation, and take the area of circle Y to be the given quantity, say 60.

Then, $\frac{16}{25}$ (60), = $\frac{16}{25}$ × 60, = ·64 × 60, = 38·4, will be the area of the inscribed square M N O P. $\frac{25}{32}$ (38·4), = $\frac{25}{32}$ × 38·4, = ·78125 × 38·4, = 30, will be the area of circle X; and is equal to half the area of circle Y. $\frac{16}{25}$ (30), = $\frac{16}{25}$ × 30, = ·64 × 30, = 19·2, will be the area of square A K L B; and is equal to half the area of square M N O P. Or, as ·78125 : 1 : : 60 : 76·8, the area of a square circumscribed about the circle Y; and it is evident that this

square is equal to four times the area of square A K L B. Therefore, $76{\cdot}8 \div 4 = 19{\cdot}2$, must be the area of square A K L B; and $\sqrt{19{\cdot}2}$ will be the value of the side of the square A K L B.

Then, A B, a side of the square A K L B, is the perpendicular of the right-angled triangle A B C, of which the value must be $\sqrt{19{\cdot}2}$. B C, the base of the triangle A B C, is, by construction, equal to three-fourths of A B. Therefore, $\frac{3}{4}(\sqrt{19{\cdot}2})$, $= \sqrt{\tfrac{9}{16} \times 19{\cdot}2}$, $= \sqrt{{\cdot}5625 \times 19{\cdot}2}$, $= \sqrt{10{\cdot}8}$, will be the value of B C, the base of the triangle A B C. And $\sqrt{AB^2 + BC^2}$, $= \sqrt{(\sqrt{19{\cdot}2}^2 + \sqrt{10{\cdot}8}^2)}$, $= \sqrt{19{\cdot}2 + 10{\cdot}8}$, $= \sqrt{30}$, will be the value of A C, the hypotheneuse of the triangle A B C.

Then, $\sqrt{19{\cdot}2}^2 = 19{\cdot}2$, will be the area of square ABFG. $\sqrt{10{\cdot}8}^2 = 10{\cdot}8$, will be the area of square B D E C. $\sqrt{30}^2 = 30$, will be the area of the square A C H I. And, $19{\cdot}2 + 10{\cdot}8 + 30$, $= 60$; and the areas of the squares ABFG, B D E C, and A C H I, are together equal to 60, the given area of circle Y.

On the orthodox data, as ${\cdot}7854 : 1 :: 60$, the given area of the circle Y, to an incommensurable quantity, equal to $76{\cdot}3942$ nearly, which will be the approximative value of the area of a square circumscribed about the circle Y. It is evident that the area of this square, is equal to four times the area of square A K L B. Therefore, $76{\cdot}3942 \div 4 = 19{\cdot}09855$, will be the approximative value of the area of square A K L B; and $\sqrt{19{\cdot}09855}$, must be the approximative value of A B, the side of square A K L B. Then, A B, is also the perpendicular of the triangle A B C, and B C, the base of the triangle A B C, is, by construction, equal to three-fourths of A B. Therefore, $\frac{3}{4}(\sqrt{19{\cdot}09855})$, $= \sqrt{\tfrac{9}{16} \times 19{\cdot}09855}$, $= \sqrt{{\cdot}5625 \times 19{\cdot}09855}$,

$= \sqrt{10{\cdot}742934375}$, will be the approximative value of B C, the base of the triangle A B C. And $\sqrt{AB^2 + BC^2}$, =

$$\sqrt{(\sqrt{19{\cdot}09855}^2 + \sqrt{10{\cdot}742934375}^2)}, =$$

$\sqrt{19{\cdot}09855 + 10{\cdot}742934375}$, $= \sqrt{29{\cdot}841484375}$; will be the approximative value of A C, the hypotheneuse of the triangle A B C.

Then, $\sqrt{19{\cdot}09855}^2 = 19{\cdot}09855$, will be the approximative value of the area of square A B F G. $\sqrt{10{\cdot}742934375}^2 = 10{\cdot}742934375$, will be the approximative value of the area of square B D F C. $\sqrt{29{\cdot}841484375}^2 = 29{\cdot}841484375$, will be the approximative value of the area of square A C H I. And, $19{\cdot}09855 + 10{\cdot}742934375 + 29{\cdot}841484375$, $= 59{\cdot}68296875$. And, the areas of the squares A B F G, B D E C, and A C H I, are together equal to $59{\cdot}68296875$, which is less than the given area of circle Y; and demonstrates that the orthodox data for the value of a circle, of which the diameter is unity, is a false data.

It has been my study to select and arrange my illustrations, as far as possible, so as to clearly elucidate my theory, and at the same time not to overburden the subject, so as to become tedious; otherwise, I might have adduced a great variety of additional demonstrations. The earnest and ingenious enquirer after truth, will find there is ample scope left him, to pursue the enquiry with interest and advantage.

It now only remains for me to deal with the latter paragraphs of your letter, and this I shall endeavour to do as succinctly as possible, for this letter has already extended beyond the limits I had originally prescribed to myself.

Some preliminary remarks are however necessary, and it is also requisite I should direct attention to some peculiar facts, bearing directly upon that portion of the sub-

ject, to which the latter clauses of your letter have special reference, before proceeding directly to reply to them.

It would appear you have at last discovered, that I am familiar with the (supposed) popular proof, consisting of the polygons inscribed in, and circumscribed to a circle; but which in reality affords no proof at all, of the relation between the diameter and circumference of a circle, as I shall presently proceed to demonstrate.

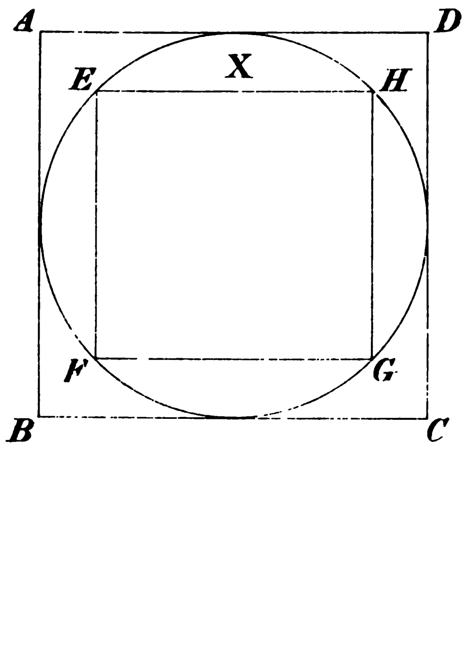
In my letter of the 13th September, I gave you a hint on this point, by directing your attention to the superficial capacity of certain geometrical figures, and to a circle equal in circumference to the earth's orbit; but, on referring to what I then stated, it strikes me I have not expressed myself in sufficiently distinct terms, and that consequently you may have misconceived my meaning; otherwise, some of your remarks would amount to a wilful perversion of my arguments. I shall endeavour on the present occasion to be more explicit, and try so to express my meaning, as to render a misconception impossible.

You will admit, that it is possible to conceive the existence of such a thing as a thin elastic line, and that such elastic line might be so arranged, as to describe a rectangle of four equal sides. Then, I put the following question:—Is it conceivable that such elastic line could be so arranged, as to describe a rectangle of any other form, and be made to contain an equal area? Your answer must be—No!

Again, it is possible to conceive that such elastic line might be so arranged, as to describe the circumference of a circle. But I may ask you:—Could it be arranged in any other form whatever, so as to be made to enclose an equal area? And again your answer must be in the negative.

FIGURE XXV.

(Diagram XIV)



To be secure of making myself intelligible, I must introduce another diagram, No. 16, (see Fig. XXV.) which simply consists of a circle X, with the square ABCD circumscribed about it, and the square EFGH inscribed in it.

This diagram is a compound figure, composed of three distinct geometrical figures, each of which contains the largest area, of which the line or lines composing its perimeter is capable in its existing form. Supposing the lines describing the perimeters of these figures to consist of thin elastic lines; it is true, the perimeters of the squares might then be changed into the form of circles, or of polygons of any number of sides, and in either case be made to contain larger areas. But the line composing the circumference of the circle X, could not be arranged in any other form whatever, and be made to contain a larger, or even an equal area.

Let the diagram of circle X be $\sqrt{32}$.

Then, $\sqrt{32}^2 = 32$, will be the area of the circumscribed square A B C D; and the area of the inscribed square E F G H, is admitted to be equal to half the area of the circumscribed square A B C D, = 16. Then, $32 \times \cdot78125 = 25$, will be the area of circle X; and it is evident, that the area of circle X, is equal to $\frac{25}{32}$ parts of the area of the circumscribed square A B C D; or, $\frac{25}{16}$ parts of the area of the inscribed square E F G H; and on this data, eight circumferences of the circle are exactly equal to twenty-five diameters.

On the orthodox data, $32 \times \cdot7854 = 25\cdot1328$, will be the area of the circle X; and on this data, the area of circle X must be equal to $\frac{25\cdot1328}{32}$ parts of the area of the circumscribed square A B C D; or, $\frac{25\cdot1328}{16}$ parts of the area of the inscribed square E F G H; and eight circum-

ferences of the circle are equal to 25·1328 diameters.*

I have already demonstrated, by means of both these data, that the area of any circle, is to the area of a square of which the perimeter is equal to the circumference of the circle; as the area of a square, to the area of an inscribed circle.

The side of the square A B C D, is equal to the diameter of circle X, $= \sqrt{32}$. Therefore, $4\ (\sqrt{32})$, $= \sqrt{4^2 \times 32}$, $= \sqrt{16 \times 32}$, $= \sqrt{512}$, must be the value of the perimeter of the square A B C D.

Let $\sqrt{512}$, represent the circumference of a circle. Then, $\sqrt{512} \div 3{\cdot}125$, $= \sqrt{512 \div \frac{25}{8}}$, $= \sqrt{512 \div \frac{625}{64}}$, $= \sqrt{512 \div 9{\cdot}765625}$, $= \sqrt{52{\cdot}4288}$, will be the diameter of the circle; $\sqrt{52{\cdot}4288}^2 = 52{\cdot}4288$, will be the area of a square circumscribed about the circle; and $52{\cdot}4288 \times {\cdot}78125$, $=$ 40·96, will be the area of the circle; and this is the value of the area of a circle, of which the circumference is equal to the perimeter of the square A B C D.

The area of the square E F G H, inscribed in the circle X, is equal to half the area of the circumscribed square A B C D, $= 16$. Therefore, $\sqrt{16} = 4$, must be the value of the side of the inscribed square E F G H; and $4 \times 4 = 16$, must be the value of the perimeter of the square E F G H.

Let 16 represent the circumference of a circle.

Then, $16 \div 3{\cdot}125 = 5{\cdot}12$, will be the diameter of the circle; $5{\cdot}12^2 = 26{\cdot}2144$, will be the area of a square circumscribed about the circle; and $26{\cdot}2144 \times {\cdot}78125$, $=$

* The advocates of the orthodox data, go no further than to say, that the arithmetical symbols ·7854 represent a very close approximation to the area of a circle of which the diameter is unity. In this illustration of my theory, I give them the advantage of supposing these figures to represent the area of the circle exactly.

20·48, will be the area of the circle; and this is the value of the area of a circle, of which the circumference is equal to the perimeter of the square E F G H.

The area of the former circle is equal to twice the area of the latter; it is evident, therefore, that the one is equal to a circle circumscribed about a square, and the other equal to a circle inscribed in such square. The value of the area of the former is 40·96, and the value of the area of the latter is 20·48.

Then, $\frac{16}{25}$ (40·96), $= \frac{16}{25} \times 40{\cdot}96$, $= {\cdot}64 \times 40{\cdot}96$, $=$ 26·2144; or, $\frac{32}{25}$ (20·48), $= \frac{32}{25} \times 20{\cdot}48$, $= 1{\cdot}28 \times 20{\cdot}48$, $=$ 26·2144; and 26·2144, will be the area of the square forming the circumscribed and inscribed square to the two circles; and agrees with the value of the square circumscribed about the smaller circle, as previously ascertained by means of the perimeter of the square E F G H.

Again, as the value of one of these numbers is equal to twice the value of the other, it is evident they may be taken to represent the areas of squares circumscribed to, and inscribed in, a circle.

Then, $\frac{25}{32}$ (40·96), $= \frac{25}{32} \times 40{\cdot}96$, $= {\cdot}78125 \times 40{\cdot}96$, $=$ 32; or, $\frac{25}{16}$ (20·48), $= \frac{25}{16} \times 20{\cdot}48$, $= 1{\cdot}5625 \times 20{\cdot}48$, $=$ 32; and 32 will be the value of the area of a circle inscribed in one, and circumscribed about the other, of such squares, and is exactly equal to the area of the square A B C D.

Again, Let $\sqrt{512}$, the value of the perimeter of the square A B C D, represent the circumference of a circle.

Then, on the orthodox data, $\sqrt{512} \div 3{\cdot}1416$, $= \sqrt{512} \div \frac{25{\cdot}1328}{8}$, $= \sqrt{512 \div \frac{631{\cdot}65763584}{64}}$, $= \sqrt{512 \div 9{\cdot}86965056}$, $= \sqrt{51{\cdot}8762}$ &c., will be the approximative value of the diameter of the circle; $\sqrt{51{\cdot}8762}^{\,2}$, $= 51{\cdot}8762$, will be the approximative value of the area of a square circumscribed

about the circle; and, 51·8762 × ·7854, = 40·74356748, will be the approximative value of the area of the circle; and on this data, is the approximative value of the area of a circle, of which the circumference is equal to the perimeter of the square A B C D.

Let 16, the value of the perimeter of the square E F G H, represent the circumference of a circle.

Then 16 ÷ 3·1416, = 5·09294 &c., will be the approximative value of the diameter of the circle; $5{\cdot}09294^2$ = 25·9380378436, will be the approximative value of the area of a square, circumscribed about the circle; and 25·9380378436 × ·7854, = 20·37173492236344, will be the approximative value of the area of the circle; and on this data, is the approximative value of the area of a circle, of which the circumference is equal to the perimeter of the square E F G H.

The area of the former circle is a close approximation to twice the area of the latter; it is evident, therefore, that the one is nearly equal in area to the area of a circle circumscribed about, and the other nearly equal in area to the area of a circle inscribed in, a square.

The value of the area of the former circle is 40·74356748, and the value of the area of the latter circle is 20·37173492236344.

Then, $\frac{16}{25{\cdot}1328}$ (40·74356748), = $\frac{16}{25{\cdot}1328}$ × 40·74356748, = ·6366182 × 40·74356748, = 25·93809659069613 6; or, $\frac{32}{25{\cdot}1328}$ (20·37173492236344), = $\frac{32}{25{\cdot}1328}$ × 20·37173492236344, = 1·273236 × 20·37173492236344, = 25·938026285610336 89184; and on this data, is a close approximation to the area of a square forming a circumscribed and inscribed square to the circles, and agrees to the fourth place of decimal figures, with the value of the square circumscribing the smaller circle, as previously

ascertained by means of the perimeter of the square E F G H.

If we take the values of these circles to represent the areas of squares circumscribed to, and inscribed in, a circle. Then, $\frac{25\cdot1328}{32}$ (40·74356748), $= \frac{25\cdot1328}{32} \times$ 40·74356748, $=$ ·7854 $\times$ 40·74356748, $=$ 31·999997898792; or, $\frac{25\cdot1328}{16}$ (20·37173492236344), $= \frac{25\cdot1328}{16} \times$ 20·37173492236344, $=$ 1·5708 $\times$ 20·37173492236344, $=$ 31·9999212160484915 52; and on this data, these figures will represent the approximative value of the area of a circle, inscribed in one, and circumscribed about the other, of such squares, and is a very close approximation to the area of the square A B C D.

The approximations in the latter case are so close, as to preclude the possibility of doubt as to the facts in the former case, and demonstrates most distinctly, that the orthodox data for the value of a circle is a false data.

In the next place, let the diameter of circle X be 4, and let a polygon of eight sides be supposed to be inscribed in the circle X. It is evident, it will be equal to a polygon of eight sides, circumscribed about the square E F G H.

Then, $4^2 = 16$, will be the area of the circumscribed square A B C D, and the area of the inscribed square E F G H, will be equal to half the area of the circumscribed square A B C D, $= 8$. Then, $\sqrt{16 \times 8}, = \sqrt{128}$, will be the area of a polygon of eight sides inscribed in circle X. But, the area of the inscribed square E F G H, is 8, and $\sqrt{8}$, must be the value of the side of the square. And, $4\,(\sqrt{8}), = \sqrt{4^2 \times 8}, = \sqrt{16 \times 8}, = \sqrt{128}$, must be the value of the perimeter of the square E F G H, inscribed in the circle X. And the values of the perimeter of the square E F G H, and the area of the polygon of eight sides, circumscribed about it, are represented by the same arithmetical symbols.

If the diameter of the circle be 8, the numerical value of the area of the inscribed polygon of eight sides, will be equal to four times the numerical value of the perimeter of the inscribed square, and as the diameter of the circle is increased, the value of the area of the inscribed polygon, as compared with the value of the perimeter of the inscribed square, increases in a geometrical progression, but the commensurable relation between the two is not destroyed.

I shall now proceed to demonstrate the fallacy, which pervades the (supposed) popular proof, consisting of the polygons inscribed in, and circumscribed to, a circle.

Let the radius of circle X be 1, and let a polygon of eight sides be supposed to be inscribed in the circle, and a polygon of eight sides circumscribed about the circle.

Then, the diameter of the circle will be 2, and $2^2 = 4$, will be the area of the square (or expressed in equivalent terms, the area of the polygon of four sides,) A B C D, circumscribed about the circle. And the area of the inscribed square (or in other words, the area of the inscribed polygon of four side) E F G H, will be equal to half the area of the circumscribed square or polygon, $= 2$.

Then, $\sqrt{4 \times 2}, = \sqrt{8}, = 2{\cdot}82842712$ &c., will be the area of an inscribed polygon of eight sides. And, as ($2 + 2{\cdot}82842712$ &c.,) : 4 : : 4 : 3·31370850 &c., the area of a circumscribed polygon of eight sides; and so far are we agreed, for there can be no doubt as to the perfect mathematical demonstrations of the facts.

But, you maintain that the mean proportional between 2·82842712 &c., and 3·31370850 &c., = 3·06146744 &c., is the area of a polygon of sixteen sides, inscribed in the circle X. And, as (2·82842712 &c., + 3·06146744 &c.,) : (2

× 2·82842712 &c.); or, as 5·88982456 &c., : 5·65685424 &c., : : 3·31370850 &c., : 3·18259790 &c.; and that these figures represent the area of a polygon of sixteen sides, circumscribed about the circle X.

This I deny; and maintain, on the contrary, that it is an assumption without the slightest proof. If I ask you for your proof, you cannot produce it, and for the best of all reasons. It is an assumption not based on fact, and a proof is therefore impossible. Running away with this false assumption, you imagine, that by pursuing the same mode of calculation, or, to adapt myself to your own phraseology, by multiplying the number of the sides of the (supposed) polygons at pleasure, they may be made to approximate to (not to equal) the circumference of the circle, and to one another, as close as we please.

Now, let us examine carefully, what this course of procedure leads to. We pursue the calculation, and ultimately arrive at what you assume to be polygons of 32·768 sides, the one inscribed in, and the other circumscribed to, the circle, and of which the values of the areas are represented by the figures 3·14159265 &c. You then reason as you were taught, without examination and without reflection, and arrive at the same conclusion as the sages who have preceded you, that the area of the circle is intermediate, between the areas of the two (supposed) circumscribed and inscribed polygons; and the circumferences of circles being as their radii, (about which there is no dispute), you jump to the conclusion, that 3·14159265 &c. must be the nearest approximation to the circumference of a circle, of which the diameter is unity, that can be obtained; and that there is no definable relation between the diameter and circumference of a circle.

In your letter of the 6th September, you make the following remarks: "*The property of one circle is the property of all circles, and if there is any circle whose radius is* 1, *and whose circumference is greater than* 3·125, *or* 100, *and greater than* 312·5, *then in all circles the ratio above named is greater than the ratio of* 1 *to* 3·125."

I am quite at one with you, when you say:—"The property of one circle is the property of all circles." And the nature of your illustration, in the paragragh from which I quote, assures me, that we are agreed in opinion as to this fact, whether the radius of a circle be one mile, 100 miles, or 10000 miles.

Now, let the radius of a circle be 10000 miles.

Then, the diameter of the circle will be 20000 miles, and $20000^2 = 400000000$ square miles, will be the area of a polygon of four sides, circumscribed about the circle. The area of a polygon of four sides inscribed in the circle, will be equal to half the area of the circumscribed polygon, $=$ 200000000 square miles. Then, $\sqrt{400000000 \times 200000000}$, $= \sqrt{80000000000000000}$, $= 282842712$ &c. square miles, will be the area of an inscribed polygon of eight sides. And, as $(200000000 + 282842712$ &c.$) : 400000000 :: 400000000 : 331370850$ &c. square miles; and these figures will represent the area of a polygon of eight sides, circumscribed about the circle. (The difference between taking the radius of the circle as one mile, or 10000 miles, consists in the *notation* of the figures, representing the values of the polygons, not in the figures themselves, for they are alike in both cases.)

Then, taking these two ascertained values, (admitted to be correct), and calculating the alternate proportionals, you imagine you obtain the areas of an inscribed and circumscribed polygon of sixteen sides, in and about the

circle; and by continuing to repeat this operation, you imagine that polygons inscribed in, and circumscribed to, the circle, are ultimately obtained, of which the areas are as nearly equal to each other as possible, and are as nearly as possible equal to the area of the circle.

The operation of obtaining alternate mean proportionals, is usually repeated until (supposed) polygons are arrived at, of 32768 sides, of which the values of the areas (the radius of the circle being 1) are represented by the figures 3·14159265 &c.

You then assume the areas of the (supposed) polygons of 32768 sides inscribed in, and circumscribed to, a circle, of which the radius is 1, to be so nearly equal to each other, that the difference between the values of the two, is expressed by about the eighth or ninth decimal place of figures; and I think you will not venture to deny, that the conclusion you draw from this is, that the area of the circle, being as you suppose intermediate between the areas of the two polygons, the perimeters of the polygons, and circumference of the circle, must also be equal, within the smallest possible fraction.

If this be true of a circle, of which the radius is one mile, it must also be true of a circle, of which the radius is 10000 miles.

Now, the radius of a circle being 10000 miles, the value of the areas of the (supposed) polygons of 32768 sides inscribed in, and circumscribed about, the circle, will be 314159265 square miles and a fraction; and is equal to the area of the circle, to the fraction of a mile in 314159265 square miles. For, the radius of the circle being 10000 miles, the diameter of it will be 20000 miles, and on the data admitted by the "authorities," 20000 + 3·14159265 &c., = 62831·853 &c., will be the circumference

of the circle. And half the circumference, multiplied by half the diameter, $= 31415 \cdot 9265$ &c., $\times 10000 = 314159265$ &c.; and 314159265 square miles and a fraction, will be the area of the circle, and is equal to the area of the (supposed) polygons of 32768 sides inscribed in, and circumscribed about it, within the fraction of a mile, in 314149265 square miles.

Now, the circumference of the circle, on the data approved by yourself, is 62831·853 &c. linear miles. Let this be divided into 32768 parts, being the number of the sides of the (supposed) polygons inscribed in, and circumscribed about, the circle. Then, 62831·853 &c., $\div$ 32768, $=$ 1·9174 &c., equal to upwards of one and nine-tenths of a mile, must be the linear value of each of these parts.

Now, it is possible to conceive that the circumference of the circle, might consist of a thin elastic line, and if so, it might be altered in form so as to describe a polygon of 32768 sides, in which case the value of each side of the polygon would be about 1·9174 linear miles, and the perimeter of the polygon would be equal to the circumference of the circle.

Then, on your assumption, you must necessarily maintain the opinion, whether the radius of the circle be 1 mile, 100 miles, or 10000 miles, that the (supposed) polygons of 32768 sides must be, the one nearly equal in area to the area of a polygon inscribed in the circle, and the other nearly equal to the area of a polygon circumscribed about the circle, and that the perimeters of the polygons, and circumference of the circle, are all equal to each other, within the smallest possible fraction.

The radius of the circle being 10000 miles, the circumference of the circle will be 62831·853 &c. linear miles, and it is evident, that the value of each side of a 32768 sided

polygon, of which the perimeter is equal to the circumference of the circle, will be equal to nearly 1·9714 parts of a linear mile.

Now, suppose it to be possible to circumscribe a circle about such a polygon, the circumference of the circle would necessarily be a line outside of, and longer than, the perimeter of the polygon, and I put the question:—How can such a polygon be inscribed in a circle of which the radius is 10000 miles, if the perimeter of the polygon, and circumference of the circle, be equal to each other within the fraction of a mile, in about 62831 linear miles?

Again, supposing it to be possible to circumscribe a circle about such a polygon, each side of the polygon would form the chord to an arc of the circle; and the area of each segment of the circle described by this chord, would be a quantity within the circle, but plus the area of the polygon. And I put the question:—How can the area of such a polygon, be equal to the area of a circle, of which the radius is 10000 miles, within the fraction of a mile, in 314159265 square miles?

Again, it is evident, that the linear value of each side of a 32768 sided polygon, of which the perimeter is equal to the circumference of a circle, whose radius is 10000 miles, is very nearly equal to two linear miles. And I put the question:—Is it not absurd to suppose, that the area of such a polygon, the area of the circle, and the area of a polygon circumscribed about the circle, can all be equal to each other, within the fraction of a mile, in 314159265 square miles?

You may probably find, should you make the attempt, that to answer these questions satisfactorily to your own mind, will prove to be a task on your ingenuity, and admitted ability, beyond the capacity of either to accomplish.

I trust I have succeeded in making my meaning intelligible; if so, and you are pleased to carefully study my arguments, in connection with those I have adduced, in my letter of the 13th September, you can hardly fail to comprehend my views, on this important branch of the subject.

A few remarks on the last paragraph of your letter, and I have done.

It would appear that to your mind, the very idea of the test of the measuring tape and a dozen stakes being fallacious, is so startling and absurd, that in your opinion any further remarks to one who could make such a statement are useless.

Now, I repeat that any such test as the measuring tape and stakes is fallacious, and I will tell you how you may prove it to be so, if you will only be at the trouble to try the experiment.

Take a disc of any diameter, it is not necessary you should know whether the diameter be 12, 16, or 18 inches, or any intermediate part of an inch. It is only necessary to take care, that the disc should be as true a circle as possible. Around it fold as tightly as you can a thin wire thread, say 16 times. This thread will form a sort of tire round the disc, something like the iron tire round a carriage wheel. Then, on a plane surface make 16 revolutions of the disc, carefully marking the distance it has run over in these 16 revolutions.

Can you imagine that the wire thread when unfolded from the disc, will describe a longer line, than the distance travelled by the disc in these 16 revolutions? I hardly suppose you can, for you must perceive that, apparently, the two should be of precisely the same length. And yet, if you will try the experiment, you will find it is so, and

the difference between the two is a distinctly appreciable quantity, about which there can be no dispute.

The reason of this may be explained. Did you ever compare the sting of a bee, with the point of a fine needle, by means of a microscope of very high power? If so, you cannot fail to have observed the striking difference between the two. Now, notwithstanding the great perfection to which mechanical skill has attained in our day, the material does not exist out of which, nor has the machinery been yet invented by means of which, a disc can be made, of which the surface shall not be as uneven, compared with a true geometrical circle, (which may be conceived but cannot be described) as the point of the finest needle, to the sting of a bee. The fact is, the most perfect disc that can be made, out of the most suitable material, and by means of the best possible machinery, is in reality a polygon of an infinite number of sides, and when you fold around it a thin wire thread, the finest the present state of mechanical skill can produce, it will form around the disc, a polygon of an infinite number of sides.

So certainly is this the fact, that I can assure you, if you will try the experiment you will find, that when you remove the wire thread from about the disc, the angles of the polygon may be detected in the thread, even with the naked eye, and also that they are very perceptible to the touch. The experiment is an interesting one, and I would recommend you to try it.

That the test of the measuring tape is fallacious, may be proved by a mechanical experiment, of another kind, if you will be at the trouble and expense of trying it, viz.: By weighing a metal disc, against a rectangle made of the same material, and supposed to contain the same superficial area as the disc. It is necessary that due care

be taken to have this experiment properly conducted, about which there can be no difficulty, if you obtain the assistance of a first class engineering mechanic.

An iron plate may be cast of any thickness, say of about one inch thick, and you may thus be assured that the material is of the same density. This iron plate may be cast in such a form, as to be most convenient to convert, one part of it into a disc, and the remaining part into two rectangles of about equal superficial area to the disc. By passing this plate through a planing machine, the surface of it may be made so even on both sides, that you may be assured it is in every part of equal thickness. I am willing to admit, that the surface of this iron plate can no more be made perfectly even, than the point of the finest needle can be made, by comparison, of equal fineness with the sting of a bee, but it can be made sufficiently true for the purposes of this experiment.

Now, you and I are perfectly agreed, that half the circumference of any circle, multiplied by half its diameter, is equal to the superficial area of the circle. Therefore, we must also be agreed in opinion, that a réctangle, of which one side is equal to the semi-circumference of such circle, and the other side equal to the semi-diameter, will contain the same superficial area as the circle.

It requires no great stretch of imagination to suppose, that you and I might be honestly and harmoniously engaged in trying this experiment, for the purpose of testing the truth of our respective opinions. Then, our first object would be, to have the disc so truly turned, that there could be no difference of opinion, as to its being as true to the circumference of a circle, as the best mechanical skill and machinery could make it. It would not be necessary, that either you or I should know the

diameter of the disc. It might be 12, 16, or 18 inches, or any intermediate and immeasurable part of an inch, our knowledge of this fact, not being essential to the experiment, as you will immediately perceive.

Then, having obtained the disc, as to the correct circumference of which, in comparison with the circumference of a circle of equal diameter, we may be supposed to be agreed, we should each, in our own way, have to produce the rectangle containing a superficial area equal to the disc.

Now, I do not assume that you would take the measuring tape for the purpose, but I presume you would take something similar to a fine wire thread, fold it as tightly as possible around the disc, and thus ascertain what you would suppose to be the exact circumference of the disc. And with a similar thread I presume you would ascertain the diameter of the disc, and dividing the two threads into two equal parts, you would then have your rectangle made, of which one side should be equal to the length of the longer of these threads, and the other side equal to the length of the shorter thread, and I imagine you to suppose, that you would thus obtain a rectangle, equal in superficial area to the superficial area of the disc.

I should adopt the plan of having a steel bar, made to the exact length of the diameter of the disc. This steel bar I should have divided and marked into sixteen equal parts, and I should order my rectangle to be made so that the longer side of it should be equal to twenty-five of these parts, and the shorter side equal to eight of such parts. This, you will observe, is in harmony with my theory. For, if the circumference of a circle be equal to three and one-eighth times its diameter, the semi-circumference will be equal to three and one-eighth times the radius or semi-diameter.

Now, I can assure you, from actual experiment, that you would find your rectangle to be heavier than the disc, and must therefore necessarily contain a larger superficial area; while on the other hand, you would find that the disc and my rectangle, would exactly balance each other, and must therefore be also equal in superficial area.

A few words more, and I shall bring this long letter to a conclusion.

You observe, that you have no theory whatever on this subject peculiar to yourself. I am not aware that I have ever said, or supposed, that you had. You have certainly endeavoured, as one having "authority" on the subject, to make that obvious to me, which, in your opinion, is obvious to every man who will carefully examine it. That your failing to do so is not your fault, I am willing to admit. That you have been at considerable trouble in your own way, to point out to me how, in your opinion I might convince myself, that it is not the world, but myself that is mistaken, I most freely acknowledge. That I regret having given you this trouble, I dare not say; and I differ from you entirely in supposing that the discussion has been of a puerile nature. To me, on the contrary, it has been both interesting and instructive, and I may tell you, that I understand the subject much better now, than I did at the commencement of our correspondence.

I can, however, honestly assure you, that I most sincerely thank you for the trouble you have taken, and if, in anything I may have said, in the course of our correspondence, I have given you the smallest annoyance, I can as honestly assure you that I sincerely regret it.

I am, Sir, yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

London, 13th December, 1860.

SIR,

I have received your letter of the 30th Nov., delivered here on the 11th December.

With reference to the words in your letter, "previously to publishing the entire correspondence," I must be allowed to make a remark.

My letters to you, were written in the sincere conviction that I was writing to one earnestly engaged in the search after truth, and my observations were confined to the pointing out to him, how he might convince himself that he was altogether wrong. My letters were not intended for publication, and I protest against their being published, for I do not wish to be gibbeted to the world as having been foolish enough to enter upon, what I feel now to have been, a ridiculous enterprize.

Therefore, I must desire that my name may not be used.

I am, Sir,
Obediently yours,
E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 15th Dec., 1860.

SIR,

I have the honour to acknowledge the receipt of your favour of the 13th inst.

You object to my publishing our correspondence on "The Quadrature of the Circle," and give your reasons for it.

You first say:—"*Your letters to me were written in the sincere conviction that you were writing to one earnestly engaged in the search after truth.*"

May I request you, to point out a paragraph throughout our correspondence, from which you can reasonably draw an opposite conclusion?

You then say, "*Your observations were confined to pointing out to me, how I might convince myself that I am altogether wrong.*"

I am willing to admit that this was your intention, and if in our correspondence, you have succeeded in accomplishing what you proposed to yourself, may I ask:—Of what have you to be ashamed if the correspondence be published?

You next say, "*Your letters were not intended for publication.*"

I can as truly say, that to a certain point in our correspondence, neither were mine.

You then protest against your letters being published, and give as your reason, "*That you do not wish to be gibbeted to the world as having been foolish enough to enter upon, what you feel now to have been, a ridiculous enterprize.*"

This sentence is capable of two interpretations. It may mean, that in an evil moment, you were foolish enough to enter into a correspondence, with a man of such ignorance and stupidity, that no amount of legitimate argument could possibly influence his silly judgment. Or, it may mean, that you undertook a task which you find yourself unable to accomplish, and that in doing so,

you were foolish enough to enter upon, what you feel now to have been, a ridiculous enterprise.

If the former be your opinion, I can see no reason why you should have the least objection to the publication of our correspondence, for in that case, I should be the party gibbeted, not you.

If the latter be your opinion, you have certainly not had the candour, to distinctly avow it.

Pray inform me which alternative you wish me to accept as your meaning, and I will then tell you what course I shall adopt, and give you my reasons for it.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 26th Jan., 1861.

SIR,

I had the honour to address you some time ago, in reply to your letter of the 13th ult., and have not since heard from you.

I now beg to wait on you with proof sheets of your two first letters. Should you not return these, I shall of course understand, that you do not wish me to send you the remainder.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

EMINENT MATHEMATICIAN, to JAMES SMITH, Esq.

Swansea, Feb. 3rd, 1861.

SIR,

The letters which you received from me, were written under an entire misconception. I had supposed that you were under the impression, that the ratio of diameter to circumference in a circle, usually adopted by practical machinists as well as theorists, was dictated by authority, and not the result of a very simple proof, learned by all beginners in trigonometry, and demonstrable by experiment. When I found that one of the proofs usually given was not accepted by you, and further, that you declined the test of experiment by measurement, I at once perceived the puerility of further discussion. What can be the object of the publication of a correspondence, originating in a mistake, and conducted under a false impression, I confess I am unable to see. The mathematical world does not want to be reminded of a familiar proof, and the non-mathematical will not understand a word.

There can be no objection on my part to the publication of the two letters you enclose me, but there may be objections to some of the others, arising from the haste in which they were written. I strongly recommend you not to publish any of them, but if you resolve otherwise, I should wish to revise the proofs.

I am, Sir,

Yours obediently,

E. M.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Wood's Hotel, Furnival's Inn, Holborn,
London, 5th Feby., 1861.

SIR,

Your letter of the 3rd Inst., has been forwarded to me here, and I have written to my publisher by this post, to request him to send you the proof sheets of your letters.

I confess I am at a loss to conceive, how your letters can have been written under a misconception on your part, as to my meaning. It appears to me, that from the commencement of our correspondence, the question at issue between us was plain and distinct, and as understood by both of us may be expressed as follows:

If the diameter of a circle be 1, you maintain that the circumference is represented by the arithmetical symbols 3·14159 &c. I maintain, on the contrary, that the figures 3·125 represent the circumference exactly; and the point at issue between us has all along been :—Which of the two (if either of them) is correct?

You are still of opinion that the data at which you arrive, is the result of a very simple proof.

Now, it is very evident, that you have not thought it worth your while to read my letter of the 30th Nov. through, or you could not have failed to observe, that I have pointed out where the fallacy lies, in what you consider this very simple proof; and also how you might convince yourself, that your proposed experiments by measurement, may be proved to be fallacious.

May I entreat you to read this letter to the end, and

I will endeavour to give you a very simple demonstration, as to the truth of my theory. It is one I have omitted to notice in the course of our correspondence.

Let the diameter of a circle be 1. Then, if a hexagon be inscribed in the circle, each side of the hexagon will be equal to the radius of the circle, and on your data, hexagonal perimeter will be to the circumference of the circle, as 3 to 3·1416. On my theory, as 3 to 3·125.

Now, if the circle be divided into 360°, it is evident that each side of the hexagon will be subtended by an arc of 60°. Then, on your data, $360° \div 3·1416 = 114°·5912$ &c., will be the *approximative* value of the diameter of the circle; $114°·5912 \div 2 = 57°·2956$, will be the *approximative* value of the radius of the circle, and equal to a side of the inscribed hexagon, which is subtended by an arc of 60°. Now, the side of the hexagon, to the arc which subtends it, must be as the perimeter of the hexagon, to the circumference of the circle; and you have admitted that if the diameter of a circle be 1, the perimeter of an inscribed hexagon is equal to 3. Therefore, as $3 : 3·1416 :: 57°·2956 : 59°·99995232$, a finite quantity, but less than the admitted value of the arc, which subtends the side of the hexagon.

On my theory, $360° \div 3·125 = 115°·2$, will be the *exact* value of the diameter of the circle; $115°·2 \div 2 = 57°·6$, will be the *exact* value of the radius of the circle, and equal to a side of the inscribed hexagon, which is subtended by an arc of 60°. Then, the side of the hexagon, is to the arc which subtends it, as the perimeter of the hexagon, to the circumference of the circle. Therefore, as $3 : 3·125 :: 57°·6 : 60°$ exactly, the admitted value of the arc; and 3·125, and no other arithmetical symbols, must represent the circumference of the circle, if the diameter of it be unity.

Now, it matters not whether we divide the circle into 360°, 400°, or any other number of degrees; or whether we hypothetically assume the circumference of the circle (the diameter being unity) to be represented by any other numbers than 3·1416, or 3·125; the fact is indisputable, that the perimeter of the hexagon is a fixed, invariable, and known quantity; and by means of which, I maintain we can arrive at a perfect demonstration, of the true circumference of the circle.

If you will only be at the trouble to examine this argument carefully, I think you cannot fail to perceive the truth of my theory: That for every linear unit contained in the diameter of a circle, there are three and one-eighth linear units contained in the circumference.

It is no doubt true, that the mathematical world does not want to be reminded of (what is supposed to be) a familiar proof, but which in reality is no proof at all; but I maintain, that any man who is a fair arithmetician and geometrician, may be made as competent to judge of the truth on this important question, as the greatest Mathematician the world ever produced; and it is of consequence that this fact should be made known.

In conclusion, I must respectfully beg to decline your advice, as to the non-publication of our correspondence.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

JAMES SMITH, Esq., to EMINENT MATHEMATICIAN.

Barkeley House, Seaforth,
Liverpool, 18th February, 1861.

SIR,

On the 11th and 12th inst., I enclosed to you some proof sheets of your letters, and hoped you would have revised and returned them without delay. As I have not heard from you, I assume you have no alterations to suggest, and I have consequently given my publisher instructions to proceed with the printing.

In my letter of the 5th inst., which was written very hurriedly, there is one point in which it occurs to me, that I have not been sufficiently explicit, to make my argument perfectly clear.

I have stated in that letter, that the perimeter of a hexagon (I should have said a regular hexagon) inscribed in a circle, of which the diameter is unity, is a fixed, invariable, and known quantity; and by means of which we can arrive at the true circumference of the circle; but I omitted to point out, that this fact may be demonstrated in several ways.

For example: Let the circumference of the circle be divided into any other number of degrees than 360°, say, 384°. Then, $384° \div 3{\cdot}125 = 122°{\cdot}88$, will be the exact value of the diameter of the circle; $122°{\cdot}88 \div 2 = 61°{\cdot}44$, will be the exact value of the radius of the circle; and equal to a side of the inscribed regular hexagon, which, in this case, is subtended by an arc of 64°. Then, the side of the hexagon, is to the arc which subtends it, as the perimeter of the hexagon, to the circumference of the

circle. Therefore, as $3 : 3{\cdot}125 :: 61^{\circ}{\cdot}44 : 64^{\circ}$, the exact value of the arc, which subtends the side of the hexagon; and it cannot be disputed that the arithmetical symbols $3{\cdot}125$, being taken to represent the circumference of a circle of which the diameter is unity, we may divide the circle into any number of degrees we like, and on this data, we can arrive at the exact value of the arc, which subtends the side of an inscribed regular hexagon.

If we assume the circumference of the circle of which the diameter is unity, to be represented by figures either less or greater in value than $3{\cdot}125$, in either case, we may approximate as close as we please to the value of the arc, which subtends the side of an inscribed regular hexagon, but in both cases fail to arrive at the exact value.

For example: Let the circumference of a circle of which the diameter is unity, be, by hypothesis, $3{\cdot}1$; and let the circumference of the circle be divided into 360°. Then, $360^{\circ} \div 3{\cdot}1 = 116^{\circ}{\cdot}129032$ &c., will be the approximative value of the diameter of the circle; $116^{\circ}{\cdot}129032$ &c. $\div 2 = 58^{\circ}{\cdot}064516$ &c., will be the approximative value of the radius of the circle; and equal to the side of an inscribed regular hexagon, which is subtended by an arc of 60°. Then, the side of the hexagon, to the are which subtends it, must be as the perimeter of the hexagon, to the circumference of the circle. Therefore, as $3 : 3{\cdot}1 :: 58^{\circ}{\cdot}064516$ &c., : $59^{\circ}{\cdot}9999998$ &c., and the decimals might be extended, *ad infinitum.*

Or, Let the circumference of a circle of which the diameter is unity, be, by hypothesis, $3{\cdot}15$; and let the circumference of the circle be divided into 360°. Then, $360^{\circ} \div 3{\cdot}15 = 114^{\circ}{\cdot}285714$ &c, will be the approximative value of the diameter of the circle; $114^{\circ}{\cdot}285714$ &c. $\div 2 = 57^{\circ}{\cdot}142857$ &c., will be the approximative value of the radius

of the circle; and equal to the side of an inscribed regular hexagon, which is subtended by an arc of 60°. Then, the side of the hexagon, is to the arc which subtends it, as the perimeter of the hexagon, to the circumference of the circle. Therefore, as 3 : 3·15 : : 57°·142857 &c. : 59°·9999985 &c., and again the decimals might be extended, *ad infinitum.*

In both examples we obtain a close approximation to the value of the arc, but in both cases, this approximation is less than its true and known value.

In the former example, we obtain a value of the radius of the circle, which is in excess of its true value; but this error is compensated by a false value of the circumference of the circle, which is less than its true value. In the latter example, the errors are reversed, and we obtain a value of the radius of the circle which is less than its true value; and the error is compensated by a false value of the circumference of the circle, which is greater than its true value. The fact is, on any false hypothetical data intermediate between 3 and 3·2, taken to represent the circumference of a circle of which the diameter is unity, we may arrive at as close an approximation as we please, to the value of the arc, which subtends the side of an inscribed regular hexagon in the circle, but never at its exact value.

You may perhaps tell me that we may assume the circumference of the circle of which the diameter is unity, to be represented by the arithmetical symbols 3 or 3·2, and in either case, by this mode of calculation, we may arrive at the symbols which represent the value of the arc, which subtends the side of an inscribed regular hexagon.

This is no doubt true; but my reply is:—To assume the former, would make the perimeter of the hexagon, and circumference of the circle; or, the side of the hexagon,

FIGURE XXVI.

(Diagram 1.)

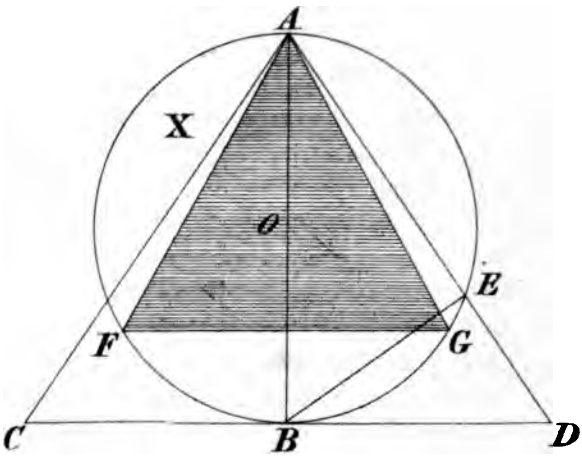
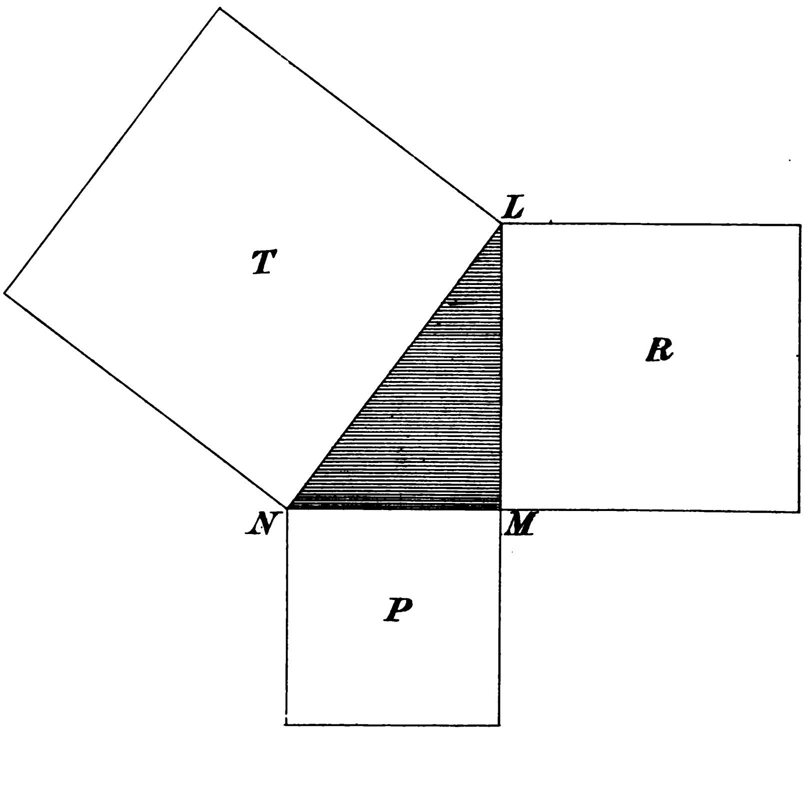


FIGURE XXVII.

(Diagram II)



and the arc which subtends it, exactly equal, which is an absurdity; and if you assume the latter, your proposed test of the measuring tape and dozen stakes, would be quite sufficient to demonstrate the fallacy; and I cannot conceive how you can escape the only remaining alternative, viz.: That 3·125, and no other arithmetical symbols, represent the true circumference of a circle, of which the diameter is unity.

If you had been an earnest and unprejudiced enquirer after truth, and had carefully examined my illustrations, and the arguments by which they are supported, you would have observed that one of the most important elements in the elucidation of my theory, is, that the commensurable right-angled triangle, of which the two sides adjacent to the right angle, are in the ratio of 3 to 4, forms a link in the chain of commensurable geometrical figures, connecting one figure with another, in a very extraordinary manner; and I shall conclude our long correspondence, by directing your attention to a very interesting illustration of this fact. It is one which I have not hitherto explicitly demonstrated, in the course of our correspondence.

For the purpose of this demonstration, the enclosed diagrams No. 1 and No. 2, (see Fig. XXVI. and Fig. XXVII.) must be taken in connection.

The following is the construction of diagram No. 1:

With the point O as centre, and with any radius, describe the circle X. Draw A B, the diameter of the circle. From the point B, draw B C, B D, at right angles to A B, making B C, B D, each equal to three-fourths of A B. Draw A C, A D, describing the right-angled triangles ABC, ABD. The line AD, cuts the circle at the point E. Join EB, describing the right-angled triangles BEA, BED.

In the circle X, inscribe the equilateral triangle A F G.

It may be demonstrated that the triangles B E A and B E D, are right-angled triangles, of which the two sides adjacent to the right angle, are in the ratio of 3 to 4.

In diagram No. 2, the squares P, R, and T, are described on the sides of the right-angled triangle L M N, of which the two sides L M, M N, adjacent to the right angle, are in the ratio of 3 to 4.

A B, the diameter of circle X, in diagram No. 1, may be any given number, say 8. Then, $8^2 = 64$, will be the area of a square described on the diameter of the circle; and $64 \times \cdot78125 = 50$, will be the area of the circle.

The side C B, is to the side A B, in the triangle A B C, in the ratio of 3 to 4, by construction. The side A B, of the triangle ABC, is the diameter of the circle X, of which the given value is 8. Therefore, $\frac{3}{4}(8) = 6$, will be the value of the side C B, of the triangle A B C; and $\frac{1}{2}(6 \times 8) = 24$, will be the superficial area of the triangle A B C.

The triangle BEA is a right-angled triangle, of which the two sides B E, A E, adjacent to the right angle, are in the ratio of 3 to 4; and A B, the diameter of the circle X, is the hypotheneuse of this triangle, and of which the given value is 8. Therefore $\frac{4}{5}(8) = 6\cdot4$, will be the value of the side A E; $\frac{3}{5}(8) = 4\cdot8$, will be the value of the side B E; and $\frac{1}{2}(6\cdot4 \times 4\cdot8) = 15\cdot36$, will be the superficial area of the triangle B E A.

The triangle BED is a right-angled triangle, of which the two sides D E, B E, adjacent to the right angle, are in the ratio of 3 to 4. And the side B E, is common to the two triangles B E A, B E D, being the longer of the two sides adjacent to the right angle, in the triangle BED, and of which the value is $4\cdot8$. Therefore, $\frac{3}{4}$ (B E), $= \frac{3}{4}(4\cdot8)$, $= 3\cdot6$, will be the value of the side D E in the triangle

B E D; and $\frac{1}{2}$ (4·8 × 3·6) = 8·64, will be the superficial area of the triangle B E D; and the triangles B E D and B E A, are together equal in superficial area, to the superficial area of the triangle ABC.

Now, let it be admitted, (for the fact is indisputable) that the diameter of the circle X being 8, the value of each side of the equilateral triangle A F G, inscribed in the circle X, is $\sqrt{48}$.

Then, let the side L M, of the triangle L M N, in diagram No. 2, be equal to a side of the equilateral triangle A F G, in diagram No. 1, $= \sqrt{48}$. The side N M, is to the side L M, in the triangle L M N, in the ratio of 3 to 4, by construction. Therefore, $\frac{3}{4}(\sqrt{48}), = \sqrt{\frac{9}{16} \times 48}, \sqrt{\cdot 5625 \times 48}, = \sqrt{27}$, will be the value of the side N M; $\frac{5}{4}(\sqrt{48},) = \sqrt{\frac{25}{16} \times 48}, = \sqrt{1 \cdot 5625 \times 48}, = \sqrt{75}$, will be the value of the side L N, in the triangle L M N; and the following results necessarily ensue:

The superficial area of square P, must be $\sqrt{27}^2 = 27$, and is exactly equal to three and one-eighth times the superficial area of the triangle B E D, in diagram No. 1. The superficial area of square R, must be $\sqrt{48}^2 = 48$, and is exactly equal to three and one-eighth times the superficial area of the triangle B E A, in diagram No. 1. The superficial area of square T, must be $\sqrt{75}^2 = 75$, and is exactly equal to three and one-eighth times the superficial area of the triangle A B C, in diagram No. 1.

Then, the area of square P, plus the area of square R, plus the area of square T, equal to $27 + 48 + 75 = 150$; and the squares P, R, and T, are together equal in superficial area, to three times the superficial area of circle X.

Again, let the diameter of a circle be equal to twice the diameter of circle X = 16. Then, $16^2 = 256$, will be

the area of a square described on the diameter of such circle; and $256 \times \cdot78125 = 200$, will be the area of the circle; and the squares P, R, and T, are together equal in superficial area, to three-fourths of the superficial area of a circle, of which the diameter is equal to twice the diameter of circle X.

These facts are analogous to, and in harmony with, the incontrovertible facts, that the superficial area of a square described on the side of an equilateral triangle inscribed in a circle, is exactly equal in superficial area to three-fourths of the superficial area of a square, described on the diameter of the circle, and three times the superficial area of a square described on its radius.

Again, let the side L M, of the triangle L M N, in diagram No. 2, be equal to the radius of circle X, in diagram No. 1, $= 4$. Then, $4^2 = 16$, will be the area of square R. $\frac{3}{4}(4) = 3$, will be the value of the side N M, of the triangle L M N; and $3^2 = 9$, will be the area of square P. $\frac{5}{4}(4) = 5$, will be the value of the side L N, of the triangle L M N; and $5^2 = 25$, will be the area of square T. And the area of square P, plus the area of square R, plus the area of square T, equal to $9 + 16 + 25 = 50$; and the squares P, R, and T, are together exactly equal in superficial area to the superficial area of circle X.

Hence:—If an equilateral triangle be inscribed in any circle, and a side of it be taken to represent the longer of the two sides of a right-angled triangle adjacent to the right angle, and of which right-angled triangle, the two sides adjacent to the right angle, are in the ratio of 3 to 4. Then, if squares be described on the sides of such right-angled triangle, the superficial area of the squares, plus the superficial area of the generating circle, are together exactly equal to the superficial area of a circle, of which

the diameter is equal to twice the diameter of the generating circle.

And :—If the radius of any circle be taken to represent the longer of the two sides of a right-angled triangle, adjacent to the right angle, and of which right-angled triangle, the two sides adjacent to the right angle, are in the ratio of 3 to 4. Then, if squares be described on the sides of such right-angled triangle, the superficial area of the squares, are together exactly equal to the superficial area of the circle.

With these illustrations I shall conclude our long correspondence. I think I have made it as obvious and evident, as that two and two make four, to anyone who will candidly and carefully examine my facts and arguments:—That for every linear unit contained in the diameter of a circle, there are three and one-eighth linear units contained in the circumference.

In conclusion, I respectfully submit, that in our correspondence, you have utterly failed to shake this proposition; and I shall be glad to find you are possessed of the moral courage to admit: That the problem of "The Quadrature of the Circle" has at length been satisfactorily solved.

I am, Sir,

Yours very respectfully,

JAMES SMITH.

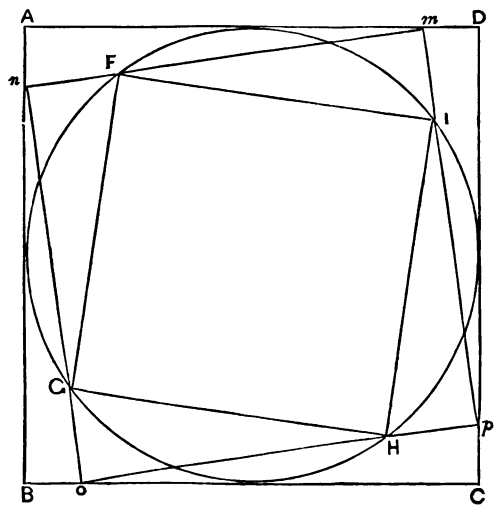
APPENDIX.

THE QUESTION:

ARE THERE ANY COMMENSURABLE RELATIONS BETWEEN A CIRCLE AND OTHER GEOMETRICAL FIGURES?

ANSWERED BY

A MEMBER OF THE "BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE."



"Strike, but Hear."

APPENDIX A.

THE QUESTION:

Are there any Commensurable Relations between a Circle and other Geometrical Figures?

TO you, Reader, I address this question. You may be a Mathematician, and so, competent to correct or to answer, as one having authority, the propositions I am about to lay before you. Or, you may belong to that larger class which accepts as matter of faith, the conclusions endorsed by the professors of this science. In either case I claim your attention;—if you are a master, to what has been wrongly taught—if you are a disciple, to what has been wrongly learned.

It has long been taught that it is impossible to demonstrate exactly the relation, in magnitude, between the Diameter and the Circumference of a Circle. I affirm that it is not impossible; and further, that I can do it. I see the smile of derision with which my affirmation is received. I have seen it very often. But it does not assure me that I am wrong. I am told that the highest authorities are against me. I know it. But this is a subject on which I

cannot admit that authority is of any value. The "highest authorities" were opposed to the proposition of Galileo. Yet he was right. Like him, I appeal to the evidence. Like him, I am told that it is all against me. And, like him, I ask my opponents how many of them have fairly examined that evidence for themselves. Very few, I believe, can answer in the affirmative. Those who have, will not stop to shelter themselves behind great names. They will know that this is a matter on which there is no room for doubt—that if I am wrong, it is not difficult to prove me so—and that whatever may be the process by which the truth is in this instance to be demonstrated, he who, even though himself in error, induces its production, aids, not retards, the progress of science. It may be as the "authorities" say it is. But if it be, let us see the proof.

Meanwhile I ask the reader's attention to the following demonstrations:—

First, as to my reasons for rejecting the data commonly accepted.

It is affirmed by authority, that the diameter of a circle is to its circumference, as 1 to 3·14159 &c., an indefinite and incommensurable relation. The figures assumed for practical purposes, to represent the nearest approximative values of the circumference and area of a circle, of which the diameter is unity, are 3·1416 for the former, and ·7854 for the latter.

Let the diameter of a circle be 4.

Then, $4 \times 3·1416 = 12·5664$, will be the circumference of the circle. Half the circumference, multiplied by half the diameter, $= \frac{1}{2}$ (12·5664) $\times \frac{1}{2}$ (4), $= 6·2832 \times 2$, $=$ 12·5664; or, the square of the diameter, multiplied by the assumed area of a circle of which the diameter is unity, $=$

$4^2 = 16$, and $16 \times \cdot 7854 = 12 \cdot 5664$, will be the area of the circle; and the circumference and area are equal in numerical value.

That the circumference and area of a circle are equal in numerical value, if the diameter of the circle be 4, may be demonstrated upon any hypothetical data, taken to represent the area of a circle of which the diameter is unity.

For example: By hypothesis, let the area of a circle of which the diameter is unity be ·785. Then, the area of a square circumscribing the circle will be unity.

The area of any square is to the area of a circle inscribed in it, as the perimeter of the square to the circumference of the circle. Therefore, as $1 : \cdot 785 :: 4 : 3 \cdot 14$; and these figures will represent the circumference of the circle.

Let the diameter of a circle be 4.

Then, on this data, $4 \times 3 \cdot 14 = 12 \cdot 56$, will be the circumference of the circle; and $\frac{1}{2}(12 \cdot 56) \times \frac{1}{2}(4), = 6 \cdot 28 \times 2, = 12 \cdot 56$; or, $4^2 = 16$, and $16 \times \cdot 785 = 12 \cdot 56$, will be the area of the circle. And the circumference and area are equal, and on any other hypothesis, a similar result would be obtained.

The attention of the reader is now directed to the following facts:—

Any hypothetical data may be taken, to represent the area of a circle inscribed in a square, of which the value of the side is unity. In this case, the area of the square and diameter of the circle will be represented by unity. If unity be divided by the figures assumed to represent the area of the inscribed circle, and the quotient be taken to represent the diameter of a circle, the value of the circumference of such circle will be a close approximation to the number 4.

For example: Let ·7854 represent the area of a circle inscribed in a square, of which the value of the side is unity. Then, 1 ÷ ·7854 = 1·273236 &c.

Let 1·273236 be taken to represent the diameter of a circle.

Then, 1·273236 × 3·1416 = 3·9999982176, will be the circumference of the circle.

Or: let ·785 represent the area of a circle inscribed in a square, of which the value of the side is unity. Then, 1 ÷ ·785 = 1·273885 &c.

Let 1·273885 represent the diameter of a circle.

Then, 1·273885 × 3·14 = 3·9999989, will be the circumference of the circle.

In both examples, the result is a close approximation to 4, as the circumference of the circle, but it will be observed that in the latter example the approximation is closer than in the former.

On the same data, there is a close approximation to numerical equality, between the diameter and area of the circle.

For example: 1 ÷ ·7854 = 1·273236 &c. Let 1·273236 represent the diameter of a circle.

Then, 1·273236 × 3·1416 = 3·9999982176, will be the circumference of the circle; $1·273236^2$ = 1·621129911696, will be the area of a square described on the diameter of the circle; and 1·621129911696 × ·7854 = 1·27323543264603B4, will be the area of the circle.

Or, 1 ÷ ·785 = 1·273885 &c. Let 1·273885 represent the diameter of a circle.

Then, 1·273885 × 3·14 = 3·9999989, will be the circumference of the circle; $1·273885^2$ = 1·622782993225, will be the area of a square described on the diameter of the circle; and 1·622782993225 × ·785 = 1273884649681625, will be the area of the circle.

In both instances there is a close approximation to numerical equality, between the diameter and area of the circle; but it will be observed, that again there is a closer approximation in the latter, than in the former example. If graduated hypothetical data, say ·784, ·783, ·782, and ·7815, be taken to represent the area of a circle of which the diameter is unity, and the results worked out, it will be found that at each step in this series of gradations, the diameter and area of the circle, approximate more nearly to equality in numerical value; and also, that in each succeeding example, there is a much closer approximation to 4, as the circumference of the circle, than in the example that precedes it.

It is sufficient to take one of these hypothetical data to illustrate this proposition, say the last, ·7815.

Then, as 1 : ·7815 : : 4 : 3·126, the circumference of the circle; and $1 \div {\cdot}7815 = 1{\cdot}2795905$ &c.

Let 1·2795905 represent the diameter of a circle.

Then, $1{\cdot}2795905 \times 3{\cdot}126 = 3{\cdot}999999903$, will be the circumference of the circle; $1{\cdot}2795905^2 = 1{\cdot}63735184769025$, will be the area of a square described on the diameter of the circle; and $1{\cdot}63735184769025 \times {\cdot}7815 =$ $1{\cdot}279590468969930375$, will be the area of the circle.

It will be observed, that in this example we obtain a closer approximation to 4, as the circumference of the circle, and also a closer approximation to numerical equality, between the diameter and area of the circle, than in the previous examples; and it appears to the writer, a reasonable inference is established, that when the true relation between the diameter and circumference of a circle is known, the number 4 being taken to represent the circumference of a circle, it will be found that the diameter and area are equal in numerical value; which is

strictly analogous to the admitted fact, that the diameter of a circle being 4, the circumference and area are equal in numerical value; and also, that a commensurable relation between the diameter and circumference of a circle will be discovered.

So much for my reasons for not accepting the Orthodox data. I will now proceed to show what I conceive to be the true solution of the problem.

Let it be admitted that the circumference of a circle is not more than three and one-seventh times, nor less than three and one-ninth times its diameter; and that the true circumference must be found somewhere between these two extremes.

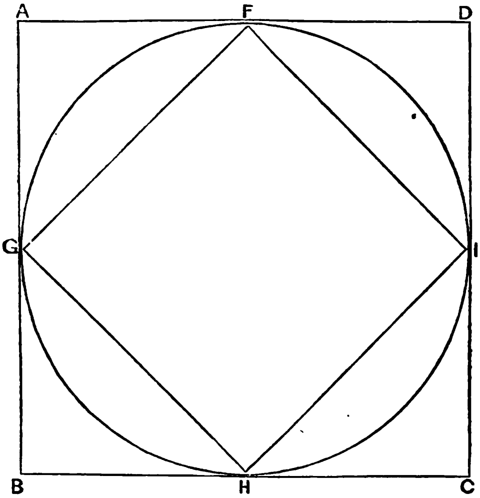
By hypothesis, let the circumference of a circle be three and one-eighth times its diameter. Then, the circumference of a circle of which the diameter is unity will be represented by the figures $3{\cdot}125$, and its area by the figures $\cdot 78125$.

Then, $1 \div \cdot 78125 = 1{\cdot}28$ exactly. And on this hypothesis, if the area of any square be divided by the area of the circle inscribed in it, the quotient will be $1{\cdot}28$, a definite, and invariable, quantity.

Let $1{\cdot}28$ represent the diameter of a circle.

Then, $1{\cdot}28 \times 3{\cdot}125 = 4$, will be the circumference of the circle. Half the circumference, multiplied by half the diameter, $= \frac{1}{2}\,(4) \times \frac{1}{2}\,(1{\cdot}28), = 2 \times \cdot 64, = 1{\cdot}28$; or, the square of the diameter, multiplied by the assumed area of a circle of which the diameter is unity, $= 1{\cdot}28^2 = 1{\cdot}6384$, and $1{\cdot}6384 \times \cdot 78125 = 1{\cdot}28$, will be the area of the circle, and the diameter and area are equal in numerical value; and realise the conditions which the writer considers essential in the solution of this question, as suggested in a previous part of this essay.

In the following diagram, let the side of square A B C D be represented by the number 4.



Then, $4^2 = 16$, will be the area of the square A B C D; $16 \times \cdot 78125 = 12 \cdot 5$, will be the area of the inscribed circle; and $4 \times 3 \cdot 125 = 12 \cdot 5$, will be the circumference of the circle.

The area of any circle is to the area of a square inscribed in it, as 25 to 16. Therefore, as 25 : 16 : : 12·5 : 8, the area of the inscribed square F G H I, and is in agreement with the admitted fact, that the area of a square circumscribing a circle, is equal to twice the area of a square inscribed in the circle.

Then:—

Area of square A B C D .	16·	
— Area of inscribed circle . .	12·5	3·5
Area of circle . . .	12·5	
— Area of inscribed square F G H I	8	4·5
Difference		1·

Therefore, this difference,

$1 \times 16 = 16$, is the area of the square A B C D.

$1 \times 12{\cdot}5 = 12{\cdot}5$, is the area of the circle.

$1 \times 50 = 50$, and 50 divided by the side of the square, $= 50 \div 4 = 12{\cdot}5$, is the circumference of the circle.

And, on the writer's hypothesis, this formula is applicable, however peculiar may be the decimal value selected as the side of the square.

For example: Let the side of square A B C D be $7{\cdot}1$.

Then:—

$7{\cdot}1^2 = 50{\cdot}41$, will be the area of square A B C D.

$50{\cdot}41 \times {\cdot}78125 = 39{\cdot}3828125$, will be the area of the circle.

As $25 : 16 :: 39{\cdot}3828125 : 25{\cdot}205$. the area of the inscribed square F G H I.

And, $7{\cdot}1 \times 3{\cdot}125 = 22{\cdot}1875$, will be the circumference of the circle.

Then:—

Area of square A B C D	50·41	
— Area of inscribed circle	39·3828125	11·0271875
Area of circle . .	39·3828125	
— Area of inscribed square F G H I . .	25·205	14·1778125
Difference		3·1506250

And:—

$3{\cdot}150625 \times 16 = 50{\cdot}41$, is the area of the square A B C D.

$3{\cdot}150625 \times 12{\cdot}5 = 39{\cdot}3828125$, is the area of the circle.

$3{\cdot}150625 \times 50 = 157{\cdot}53125$, and $157{\cdot}53125 \div 7{\cdot}1 = 22{\cdot}1875$, is the circumference of the circle.

Again: If the area of square A B C D be unity, on the writer's hypothesis, the figures ·78125, will represent the area of the circle inscribed in it.

Then :—

Area of square A B C D	.	1·	
— Area of inscribed circle	.	·78125	·21875
Area of circle	. . .	·78125	
— Area of inscribed square F G H I		·5	·28125
Difference			·06250

And :—

·0625 × 16 = 1, is the area of the square A B C D.

·0625 × 12·5 = ·78125, is the area of the circle.

·0625 × 50 = 3·125, and 3·125 ÷ 1 = 3·125, is the circumference of the circle.

By hypothesis, let the area of a circle of which the diameter is unity, be ·78124, and apply the given formula.

Then :—

Area of square A B C D	.	1·	
— Area of inscribed circle	.	·78124	·21876
Area of circle	. . .	·78124	
— Area of inscribed square F G H I		·5	·28124
Difference			·06248

And :—

·06248 × 16 = ·99968. But these figures make the area of the square A B C D too little.

By hypothesis, let the figures ·78126 represent the area of a circle, of which the diameter is unity.

Then :—

Area of square A B C D	.	1·	
— Area of inscribed circle	.	·78126	·21874
Area of circle	. . .	·78126	
— Area of inscribed square F G H I		·5	·28126
	Difference		·06252

And :—

·06252 × 16 = 1·00032. But these figures are in excess of the admitted value of the area of the square A B C D, and it appears to the writer impossible to avoid the conclusion, that ·78125, and no other figures, represent the true area of a circle, of which the diameter is unity.

A similar formula may be applied to the cube.

For example : Let the side of a cube be represented by the number 7.

Then :—

$7^3 = 343$, will be the cubical contents of the cube.

343 × ·78125 = 267·96875, will be the cubical contents of a sphere, of which the diameter is 7, and equal to the side of the cube.

As, 25 : 16 : : 267·96875 : 171·5, the value of half the cube.

Then :—

Cubical contents of cube		343·	
— Cubical contents of sphere		267·96875	75·03125
Contents of sphere	.	267·96875	
— ½ contents of cube	.	171·5	96·46875
	Difference		21·43750

And :—

$21{\cdot}4375 \times 16 = 343$, is the cubical contents of the cube.

$21{\cdot}4375 \times 12{\cdot}5 = 267{\cdot}96875$, is the cubical contents of the sphere.

$21{\cdot}4375 \times 50 = 107{\cdot}1875$; and these figures divided by the diameter of the sphere squared, or, $107{\cdot}1875 \div 7^2 = 107{\cdot}1875 \div 49 = 21{\cdot}875$, is the circumference of the sphere, and is equal to three and one-eighth times its diameter.

In treating of the relations between a circle and a square, we have proved, on the writer's hypothesis, that, if the circumference of a circle be represented by the number 4, the diameter and area of the circle are equal in numerical value, and are represented by the figures 1·28.

Let 1·28 be taken to represent the side of a cube.

Then :—

$1{\cdot}28^3 = 2{\cdot}097152$, will be the contents of the cube.

$2{\cdot}097152 \times {\cdot}78125 = 1{\cdot}6384$, will be the contents of the sphere.

As, $25 : 16 :: 1{\cdot}6384 : 1{\cdot}048576$, will be half the contents of the cube.

And, $1{\cdot}28 \times 3{\cdot}125 = 4$, will be the circumference of the sphere.

And :—

Contents of cube	.	2·097152	
— Contents of sphere	.	1·6384	·458752
Contents of sphere	.	1·6384	
— ½ contents of cube	.	1·048576	·589824
	Difference		·131072

And :—

$\cdot 131072 \times 16 = 2\cdot 097152$, is the contents of the cube.

$\cdot 131072 \times 12\cdot 5 = 1\cdot 6384$, is the contents of the sphere.

$\cdot 131072 \times 50 = 6\cdot 5536$; and these figures divided by the diameter of the sphere squared; or,

$6\cdot 5536 \div 1\cdot 28^2, = 6\cdot 5536 \div 1\cdot 6384, = 4$, is the circumference of the sphere, and is equal to three and one-eighth times its diameter.

The reader will observe, that the square of the diameter of the sphere, and the cubical contents of the sphere, are, in this instance, equal in numerical value, and are represented by the figures 1·6384; and his attention is directed to the following remarkable coincidence :—

The cubical contents of the sphere, and the square of the diameter of the sphere, are 1·6384.

Then :—

$1\cdot 6384^2 = 2\cdot 68435456$; and if these figures be multiplied by ·78125, the area of a circle of which the diameter is unity, on the writer's hypothesis, the product is equal to the contents of the cube. For, $2\cdot 68435456 \times \cdot 78125 = 2\cdot 097152$, is the contents of the cube.

The relations between the figures composing the diagram, may be illustrated in the following manner :—

Let the diameter of the circle be the given quantity, and let its value be represented by the number 8.

Then, on the writer's hypothesis :—

$8^2 = 64$, will be the area of the circumscribing square A B C D.

$8 \times 3\cdot 125 = 25$, will be the circumference of the circle.

$64 \times \cdot 78125 = 50$, will be the area of the circle.

As, 25 : 16 : : 50 : 32, the area of the inscribed square F G H I.

Then :—

The area of the circumscribing square A B C D, divided by the circumference of the circle, = $64 \div 25 = 2{\cdot}56$.

The area of the circle divided by the area of the inscribed square F G H I, $= 50 \div 32 = 1{\cdot}5625$.

And, $2{\cdot}56 \times 1{\cdot}5625 = 4$, is the radius of the circle.

In the next place, let the circumference of the circle be the given quantity, and let it be represented by the number 60.

Then :—

$60 \div 3{\cdot}125 = 19{\cdot}2$, will be the diameter of the circle.

$19{\cdot}2^2 = 368{\cdot}64$, will be the area of the circumscribing square A B C D.

$368{\cdot}64 \times {\cdot}78125 = 288$, will be the area of the circle.

As, 25 : 16 : : 288 : 184·32, the area of the inscribed square F G H I.

Then :—

The area of the circumscribing square A B C D, divided by the circumference of the circle, = $368{\cdot}64 \div 60 = 6{\cdot}144$.

The area of the circle, divided by the area of the inscribed square F G H I, $= 288 \div 184{\cdot}32 = 1{\cdot}5625$.

And, $6{\cdot}144 \times 1{\cdot}5625 = 9{\cdot}6$, is the radius of the circle.

Again : Let the area of the circle be the given quantity, say 60.

Then :—

As, ·78125 : 1 : : 60 : 76·8, the area of the circumscribing square A B C D.

As, 25 : 16 : : 60 : 38·4, the area of the inscribed square F G H I.

The $\sqrt{76 \cdot 8}$ = 8·76356092 &c., an incommensurable quantity, will be the approximative value of the diameter of the circle.

And, 8·76356092 &c. × 3·125 = 27·3861127875 &c., will be the approximative value of the circumference of the circle.

Then :—

The area of the circumscribing square A B C D, divided by the circumference of the circle, = 76·8 ÷ 27·3861127875 = 2·804339494.

The area of the circle, divided by the area of the inscribed square F G H I, = 60 ÷ 38·4 = 1·5625.

And, 2·804339494 × 1·5625 = 4·38178046, is the approximative value of the radius of the circle, and is equal to half the diameter, worked out to eight decimal places, and would continue to be so, if the decimals were extended, *ad infinitum.*

It is thus demonstrated, that if 3·125 be taken to represent the circumference, and ·78125 to represent the area of a circle, of which the diameter is unity, the diameter, circumference, or area of the circle, may be the given quantity, and by the above formula the radius of the circle, a known and admitted quantity, may be arrived at. And it again appears to the writer, to be an impossibility to avoid the conclusion, that the figures 3·125 represent the true circumference, and ·78125 the true area of a circle, of which the diameter is unity.

In addition to the methods adopted for ascertaining

the values of the areas of the figures composing the diagram, the following rule may likewise be applied:—

Let the diameter of the circle, and the side of square A B C D, be 8.

Then:—

$8^2 = 64$, will be the area of the circumscribing square A B C D.

$64 \div 1{\cdot}28 = 50$, will be the area of the circle.

$50 \div 1{\cdot}5625 = 32$, will be the area of the inscribed square F G H I.

And, $1{\cdot}28 \times 1{\cdot}5625 = 2$, is the square root of 4, the first square of which the values of the area and its root can be expressed in integers, or whole numbers; and these numbers are repeatedly introduced to the notice of the reader in the course of this inquiry.

The following methods may be adopted for ascertaining the circumference of the circle:—

Let the diameter of a circle be 7.

The diameter of any circle is to its circumference, as 1·28 to 4.

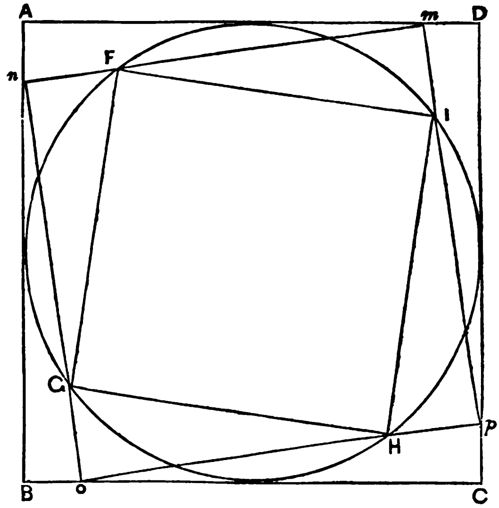
Then:—

As $1{\cdot}28 : 4 :: 7 : 21{\cdot}875$, the circumference of the circle.

Or, if five-eighths of the diameter of any circle be multiplied by 5, the product will be the circumference of the circle. For, $\frac{5}{8}(7) = 4{\cdot}375$, and $4{\cdot}375 \times 5 = 21{\cdot}875$, is the circumference of the circle.

And both methods are in harmony with the writer's hypothesis. For, $7 \times 3{\cdot}125 = 21{\cdot}875$, is the circumference of the circle.

The attention of the reader is now requested to the following diagram :—



Let A B C D be a square, with a circle inscribed in it. From each side of the square cut off a part, A*n*, B*o*, C*p*, D*m*, equal to one-eighth. Draw *m n*, *n o*, *o p*, *p m*, describing the square *m n o p*, and the right-angled triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*. The sides of the square *mnop* cut the circle at the points F G H I. (Each side of the square *m n o p* cuts the circle in two points: either or both of them might be taken, but for the purpose of demonstration it is sufficient to take one.) Draw F G, G H, H I, I F, describing the square F G H I, and the right-angled triangles F *n* G, G *o* H, H *p* I, I *m* F. The square F G H I is an inscribed square in the circle, and admitted to be equal in value to half the circumscribing square A B C D.

On the writer's hypothesis, the circle and the square *m n o p* are equal.

Let the side of the square A B C D, and the diameter of

the circle, be represented by the number 8.

Then:—

$8^2 = 64$, will be the area of square A B C D: and,

$64 \times \cdot 78125 = 50$, will be the area of the circle.

The triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*, are right-angled triangles, and are all equal, and each is equal to half a rectangle, of which the value of the longer side is 7, and the value of the shorter side 1. Therefore, $\frac{1}{2}(7 \times 1) = 3\cdot5$, will be the value of the area of each triangle; and, $3\cdot5 \times 4 = 14$, will be the value of the area of the four triangles.

Then:—

Area of square A B C D . . .	64·
— Area of the 4 triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*	14
Remainder	50

And this remainder will be area of the square *mnop*, and is equal in value to the area of the circle.

Again: Each of the triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*, is a right-angled triangle, of which the value of the sides adjacent to the right angle is 7 and 1. Therefore, $\sqrt{7^2 + 1^2}, = \sqrt{49 + 1}, = \sqrt{50}$, will be the value of the hypotheneuse.

The hypotheneuse of the triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*, together make up the square *mnop*. Therefore, the side of square *m n o p* must be $\sqrt{50}$, and $\sqrt{50}^2 = 50$, will be the value of its area; and it is again demonstrated that the area of square *m n o p*, and the area of the circle, are equal.

Again: The four angles of the square A B C D, cut off by the circumference of the circle, are equal in value to the four triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*. And it has been demonstrated that the area of the circle, and the area of square *m n o p*, are equal, and that the value of each is 50.

Therefore :—

	Area of square A B C D	64
—	Area of circle	50
	Remainder	14

And this remainder will be the value of the four angles of the square A B C D, cut off by the circumference of the circle.

And :—

	Area of square A B C D	64
—	Area of square *m n o p*	50
	Remainder	14

And this remainder will be the value of the four triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*.

But these remainders are equal; therefore, the four angles of the square A B C D, cut off by the circumference of the circle, are equal in value to the four triangles *m* A *n*, *n* B *o*, *o* C *p*, *p* D *m*, of which the sides of the square *m n o p* form the hypotheneuse, each to each.

Again: The four segments of the circle, of which the sides of the inscribed square F G H I describe the chords, are equal in value to the four triangles F *n* G, G *o* H, H *p* I, I *m* F, of which the sides of the square F G H I form the hypotheneuse.

The area of the circle and the area of square *m n o p* are equal, and the value of each is 50. And the area of square F G H I is 32.

Therefore :—

	Area of circle	50
—	Area of square F G H I	32
	Remainder	18

And this remainder will be the value of the four

segments of the circle cut off by the sides of the inscribed square F G H I.

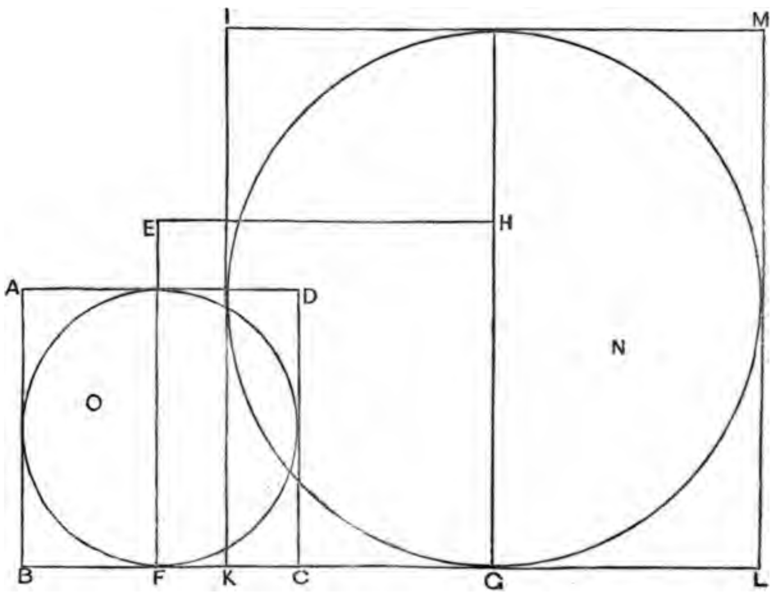
And :—

Area of square *m n o p*	. . .	50
— Area of square F G H I	. . .	32
	Remainder	18

And this remainder will be the value of the four triangles F *n* G, G *o* H, H *p* I, I *m* F.

But these remainders are equal; therefore, the four segments of the circle cut off by the sides of the inscribed square F G H I, are equal to the four triangles F *n* G, G *o* H, H *p* I, I *n* F, of which the sides of the square F G H I form the hypotheneuse, each to each.

The reader's attention is now directed to the following diagram :—



By construction, let the side of square A B C D, and the diameter of circle O, be to the side of square E F G H in the ratio of 4 to 5; and the side of square E F G H to the side of square I K L M, and diameter of circle N, in the ratio of 5 to 8. Then, if the side of square A B C D and diameter of circle O be 4, the side of square E F G H will be 5, and the side of square I K L M, and the diameter of circle N, will be 8. It will be admitted that the side of square I K L M, being twice the value of the side of square A B C D, the area of square I K L M, will be four times the area of square A B C D; and the diameter of circle N, being twice the value of the diameter of circle O, the area of circle N, will be four times the area of circle O.

Then :—

The side of square I K L M is 8, and $8^2 = 64$, will be the area of square I K L M.

The side of square E F G H is 5, and $5^2 = 25$, will be the area of square E F G H.

The side of square A B C D is 4, and $4^2 = 16$, will be the area of square A B C D.

Then: The area of the square I K L M, divided by the area of the square E F G H, $= 64 \div 25 = 2{\cdot}56$; and, on the writer's hypothesis, $\sqrt{2{\cdot}56}$ is the diameter of a circle, of which the area is 2. The area of the square E F G H, divided by the area of the square A B C D, $= 25 \div 16 = 1{\cdot}5625$; and $1{\cdot}5625$ is the area of a circle, of which the diameter is $\sqrt{2}$. $2{\cdot}56 \times 1{\cdot}5625 = 4$, is the diameter of circle O, and radius of circle N. And the mean proportional between $2{\cdot}56$ and $1{\cdot}5625$ is 2. For, $\sqrt{2{\cdot}56 \times 1{\cdot}5625} = \sqrt{4} = 2$. And the importance ot these facts, will be observed in the demonstrations by means of this diagram.

The area of square A B C D is 16, and $16 \times .78125 = 12.5$, will be the area of circle O.

The area of square I K L M is 64, and $64 \times .78125 = 50$, will be the area of circle N.

The area of square EFGH is 25, an admitted quantity.

And, on the writer's hypothesis, it is demonstrated that the area of circle O is equal to half the area of square E F G H; and the area of circle N is equal to twice the area of square E F G H, and four times the area of circle O; and that there is, consequently, a commensurable relation between the circles N and O, and the square E F G H.

Let the diameter of circle O be 7.

Then, $7^2 = 49$, will be the area of the circumscribing square A B C D; and $49 \times .78125 = 38.28125$, will be the area of circle O.

The diameter of a circle is to the side of a square containing twice the area of the circle, in the ratio of 4 to 5. Therefore, as $4 : 5 :: 7 : 8.75$; and $8.75^2 = 76.5625$, will be area of square E F G H, and is equal to twice the area of circle O. For, the area of circle O is 38.28125, and $38.28125 \times 2 = 76.5625$, is the area of square E F G H.

Again: Let the side of square E F G H be 9.

Then, $9^2 = 81$, will be the area of the square.

The side of a square is to the diameter of a circle containing twice the area of the square, in the ratio of 5 to 8. Therefore, as $5 : 8 :: 9 : 14.4$, the diameter of circle N, and side of square I K L M; $14.4^2 = 207.36$, will be area of the circumscribing square I K L M; and $207.36 \times .78125 = 162$, will be area of circle N, and is equal to twice the area of square E F G H. For, the area of square E F G H is 81, and $81 \times 2 = 162$, is the area of circle N.

Let the side of square E F G H be $\sqrt{7}$, and to the side of square I K L M, or diameter of circle N, in the ratio of 5 to 8. Required, the value of the side of square I K L M, or diameter of circle N, in perfect decimal expression.

Then :—

> $7 \times 2{\cdot}56 = 17{\cdot}92$, and $\sqrt{17{\cdot}92}$ is the required number.

For :—

> $\sqrt{7} = 2{\cdot}645751311064$ &c. $\sqrt{17{\cdot}92} = 4{\cdot}2332020977$ &c.; and, as $5 : 8 :: 2{\cdot}645751311064 : 4{\cdot}2332020977$, and the decimals might be continued, *ad infinitum.*

Again, let the side of square A B C D, and diameter of circle O, be to the side of square E F G H in the ratio of 4 to 5, and let the side of square E F G H be $\sqrt{7}$.

Required, the side of square A B C D, and diameter of circle O, in perfect decimal expression.

Then :—

> $7 \div 1{\cdot}5625 = 4{\cdot}48$, and $\sqrt{4{\cdot}48}$ is the required number.

For :—

> $\sqrt{4{\cdot}48} = 2{\cdot}11660104885$ &c. $\sqrt{7} = 2{\cdot}64575131106$ &c.; and, as $4 : 5 :: 2{\cdot}11660104885 : 2{\cdot}64575131106$, and the decimals might be continued, *ad infinitum.*

It has been demonstrated, on the writer's hypothesis, that if the diameter of a circle be to the side of a square in the ratio of 4 to 5, the area of the square is twice the area of the circle; and that if the side of a square be to the diameter of a circle in the ratio of 5 to 8, the area of the circle is twice the area of the square, and the attention of the reader may also be directed to the following analogous relations between a circle and a square.

If the circumference of a circle be divided by 5, the quotient will be the side of a square, of which the area is equal to half the area of the circle.

Let the circumference of a circle be 10·5.

Then :—

$10{\cdot}5 \div 3{\cdot}125 = 3{\cdot}36$, will be the diameter of the circle.

$3{\cdot}36^2 = 11{\cdot}2896$, will be the area of a square circumscribed about the circle.

And, $11{\cdot}2896 \times {\cdot}78125 = 8{\cdot}82$, will be the area of the circle.

Let $10{\cdot}5 \div 5 = 2{\cdot}1$, be the side of a square.

Then :—

$2{\cdot}1^2 = 4{\cdot}41$, will be the area of the square, and is equal to half the area of the circle.

Again: If the value of the four sides of a square be divided by 5, the quotient will be the diameter of a circle of which the area is equal to half the area of the square.

Let the value of the four sides of a square be 6.

Then :—

$6 \div 4 = 1{\cdot}5$, will be the side of the square.

$1{\cdot}5^2 = 2{\cdot}25$, will be the area of the square.

And :—

$6 \div 5 = 1{\cdot}2$, will be the diameter of the circle.

$1{\cdot}2^2 = 1{\cdot}44$, will be the area of a square circumscribed about the circle.

And, $1{\cdot}44 \times {\cdot}78125 = 1{\cdot}125$, will be the area of the circle, and is equal to half the area of the square.

The reader's attention may be here directed to some peculiar relations, touching commensurable right-angled triangles, as introductory to the consideration of the next diagram.

Let A and B represent any two consecutive numbers.

Then, the sum of the two numbers will be one side, twice the product of the two nnmbers will be another side, and twice their product, plus one, will be the third side of a commensurable right-angled triangle.

For example: Let A represent 1, and B represent 2. Then:—

The sum of A B, $= 1 + 2 = 3$, will be one side.

Twice the product of A B,$= 1 \times 2 \times 2 = 4$, will be another side.

Twice the product of A B, plus one, $= 4 + 1 = 5$; which is equal to $\sqrt{3^2 + 4^2}$,$=$ $\sqrt{9 + 16}$,$=$ $\sqrt{25}$, $= 5$, will be the third side.

With this formula, 2 and 3 will produce		5	12	13.
3 and 4	„	7	24	25.
4 and 5	„	9	40	41.
5 and 6	„	11	60	61.
6 and 7	„	13	84	81.
7 and 8	„	15	112	113.
8 and 9	„	17	144	145.

And a peculiar commensurable relation is established between triangles as thus constructed.

The two shorter sides of any right-angled triangle are the two sides adjacent to the right angle; and with reference to any of the above triangles, the two sides adjacent to the right angle, plus the shortest side of the following triangle, are together equal to the other side adjacent to the right angle, of such following triangle.

For:—

$$5 + 12 + 7 = 24.$$
$$7 + 24 + 9 = 40.$$
$$9 + 40 + 11 = 60.$$
$$11 + 60 + 13 = 84.$$

$13 + 84 + 15 = 112.$

$15 + 112 + 17 = 144.$ And so we might proceed, *ad infinitum.*

Again: A and B may be taken to represent any two numbers, of which let B be the greater.

Then:—

Twice the product of the two numbers will be one side.

The square of the greater, minus the square of the lesser, will be another side.

The square of the greater, plus the square of the lesser, will be the third side of a commensurable right-angled triangle.

For example: Let A represent 2, and B represent 3.

Then:—

Twice the product of A B, $= 2 \times 3 \times 2 = 12$, will be one side.

$B^2 - A^2, = 3^2 - 2^2, = 9 - 4, = 5$, will be another side.

$B^2 + A^2, = 3^2 + 2^2, = 9 + 4, = 13$, will be the third side of a commensurable right-angled triangle. For, $\sqrt{12^2 + 5^2}, = \sqrt{144 + 25} = \sqrt{169} = 13.$

Let A and B represent 5 and 12, the value of the two lesser sides of this triangle, and be the given numbers to find another commensurable right-angled triangle.

Then:—

Twice the product of A B, $= 5 \times 12 \times 2 = 120$, will be one side.

$B^2 - A^2, = 12^2 - 5^2, = 144 - 25, = 119$, will be another side.

$B^2 + A^2, = 12^2 + 5^2, = 144 + 25, = 169$, will be the third side; and the third side of this triangle

is equal to the square of the third side of the triangle in the preceding example, and so we might proceed, *ad infinitum.*

Again: Let ·6 and ·8 represent the two sides of a right-angled triangle, adjacent to the right angle.

Then :—

$\sqrt{\cdot 6^2 + \cdot 8^2}$, $= \sqrt{\cdot 36 + \cdot 64}$, $= \sqrt{1}$, $=$ unity, will be the value of the third side of the triangle.

Let A represent ·6, and B ·8, and let these be the two given numbers, to find another commensurable right-angled triangle.

Then :—

$2\,A\,B$, $= \cdot 6 \times \cdot 8 \times 2 = \cdot 96$, will be one side.

$B^2 - A^2$, $= \cdot 8^2 - \cdot 6^2$, $= \cdot 64 - \cdot 36$, $= \cdot 28$, will be another side.

$B^2 + A^2$, $= \cdot 8^2 + \cdot 6^2$, $= \cdot 64 + \cdot 36$, $=$ unity; Or, $\cdot 96^2 + \cdot 28^2$, $= \cdot 9216 + \cdot 0784$, $=$ unity, will be the third side of the triangle.

Again: If we take ·28 and ·96 to represent A and B,

Then :—

$2\,A\,B$, $= \cdot 28 \times \cdot 96 \times 2 = \cdot 5376$, will be one side.

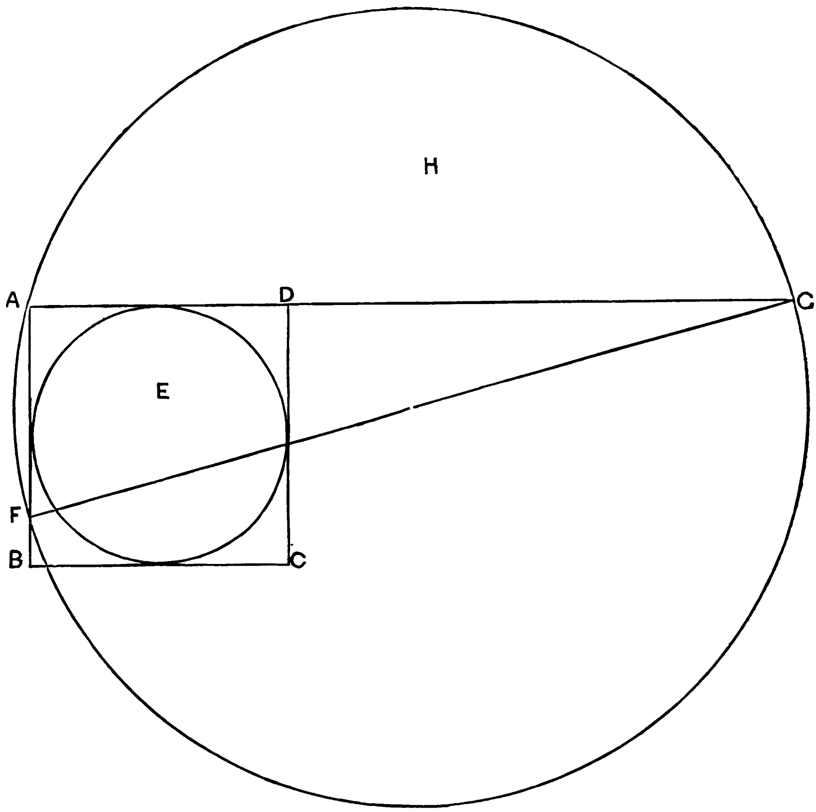
$B^2 - A^2$, $= \cdot 96^2 - \cdot 28^2$, $= \cdot 9216 - \cdot 0784$, $= \cdot 8432$, will be another side.

$B^2 + A^2$, $= \cdot 96^2 + \cdot 28^2$, $= \cdot 9216 + \cdot 0784$, $=$ unity; Or, $\cdot 5376^2 + \cdot 8432^2$, $= \cdot 28901376 + \cdot 71098624$, $=$ unity, will be the third side of the triangle.

And so we might proceed, *ad infinitum,* and the third side of the triangle would continue to be unity.

If 3 and 4 be the given numbers to find a commensurable right-angled triangle, the value of the three sides of such triangle will be 7, 24, and 25, respectively, and

the reader is requested to bear this in mind in the consideration of the following diagram :—



On the straight line A B, describe the square A B C D, and inscribe in it the circle E. From A B cut off a part, A F, equal to seven-eighths of A B; or F B, equal to one-eighth of A B. Prolong A D to G, making D G equal to twice the side of square A B C D, and draw F G. With F G as diameter, describe the circle H.

Let the value of the side of square A B C D be represented by the number 8. Then the three sides of the triangle G A F, will be represented by the figures 7, 24, and 25, and the triangle will represent the commensurable right-angled triangle, derived from the two given numbers 3 and 4.

The side A B of the square A B C D is, by construction, 8; and the part F B cut off, is, by construction, an eighth part of it, or 1.

Therefore :—

A B — F B, = 8 — 1, = 7, will be the value of the side A F of the triangle G A F.

A D + D G will be the value of the side A G of the triangle G A F. But, D G is, by construction, equal to twice the side of square ABCD, and A D is one side of the square. Therefore, the side A G, of the triangle G A F, is equal to three times the side of square A B C D, = $8 \times 3 = 24$.

The angle A is a right angle, and G A, A F, are the two sides of the triangle adjacent to the right angle. Therefore, $\sqrt{A\ G^2 + A\ F^2}$, = $\sqrt{24^2 + 7^2}$, = $\sqrt{576 + 49}$, = $\sqrt{625}$, = 25, will be the value of the hypotheneuse, or third side of the triangle G A F, and is equal to the diameter of circle H.

The diameter of circle H is 25, and if a square were

circumscribed about it, the side of the square would be 25.

Let the square A B C D, be supposed to represent such square. The diameter of circle E will be equal to the side of the square A B C D.

Then :—

A B — F B, will be equal to $25 - 3{\cdot}125 = 21{\cdot}875$, and these figures will represent the side A F, of the triangle G A F.

The side A G, of the triangle G A F, is, by construction, equal to three times the side of square A B C D; therefore, $25 \times 3 = 75$, will represent the value of the side A G of the triangle G A F.

And, $\sqrt{AF^2 + AG^2}, = \sqrt{21{\cdot}875^2 + 75^2}, = \sqrt{478{\cdot}515625 + 5625}, = \sqrt{6103{\cdot}515625}, = 78{\cdot}125$, will be the value of F G, the third side of the triangle G A F, and on the writer's hypothesis is equal to the circumference of circle E. For, 25 is the diameter of circle E, and $25 \times 3{\cdot}125 = \underline{78{\cdot}125}$, will be the circumference of circle E.

The reader will observe that, in the latter example, every line is three and one-eighth times greater than the relative line in the former example; and this must necessarily be so, whatever figures may be assumed to represent the circumference and area of a circle.

On the writer's hypothesis, however, there is a commensurable relation between the circumference of the circle E, and the hypotheneuse of the triangle G A F, for they are equal; while on the Orthodox data, the relation between the two is just as indefinite as that which is supposed to exist between the diameter and circumference of a circle.

To the writer's mind this evidence is conclusive, and he does not doubt that it will be equally so to any one

who will give the subject a candid and careful consideration, that the circumference of a circle is, in fact, neither more nor less than three and one-eighth times its diameter.

It is only necessary for the writer's present purpose to give one more illustration.

Let the diameter of a circle be 768.

On the Orthodox data, $768 \times 3{\cdot}1416 = 2412{\cdot}7488$, will be the circumference of the circle.

On the writer's hypothesis, $768 \times 3{\cdot}125 = 2400$, will be the circumference of the circle.

By hypothesis, let the circumference of the circle be equal to the perimeter of a polygon of 40000 sides.

Then, on the former data, $2412{\cdot}7488 \div 40000 = {\cdot}06031872$.

On the latter data, $2400 \div 40000 = {\cdot}06$.

If 40000 equidistant radii be supposed to be drawn in the circle, the value of that part of the circumference of the circle contained by any two radii, will be represented by one or other of the above sets of figures, as one or other of the assumed data may be adopted.

It is quite certain that no subdivision of the circumference of a circle, however minute, can resolve that part of it contained by any two radii into a straight line; but for the purpose of this demonstration, it is sufficient that it be admitted, that the part of the circle contained by two radii, if the subdivision be very minute, will describe a figure very nearly equal to an isosceles triangle of which the two sides are equal to the radius of the circle, and the base equal to that part of the circumference of the circle contained by any two radii. The radius of the circle, in this instance, is 384; a definite and admitted quantity.

And if the part of the circle contained by two radii

be supposed to represent a figure very nearly equal to an isosceles triangle, of which the value of the sides is 384, the data, though only hypothetical, by which we can most nearly approach these figures (an admitted quantity), must be (if not true) the nearest approximation to the truth.

It will be admitted that, on either data, the sides of the (supposed) isosceles triangle will be equal to a radius of the circle, and will be represented by the number 384.

On the Orthodox data, the base of the triangle will be ·06031872. On the writer's hypothesis, the base of the triangle will be ·06.

Then:—

> If the angle of the triangle be bisected, and a line drawn from it to the base, we obtain two right-angled triangles, of which the value of the side subtending the right angle will be 384, the admitted value of the radius of the circle, and $\frac{1}{2}$ (·06031872) = ·03015936, or, $\frac{1}{2}$ (·06) = ·03, will be the base, as one or other of these contrasted data shall be adopted.

Then :—

> $384^2 - \cdot03015936^2, = 147456 - \cdot0009095869956096,$ $= 147455\cdot9990904130043904$, the square root of which is $383\cdot9999988815$, will be the value of the third side.

Or :—

> $384^2 - \cdot03^2, = 147456 - 0009, = 147455\cdot9991$, the square root of which is $383\cdot999998828$, will be the value of the third side.

If, on the other hand, we take the bisecting line of the (supposed) isosceles triangle (admitted to be equal to the radius) and the base, to be the two sides of a triangle adjacent to the right angle.

Then :—

$384^2 + {\cdot}03015936^2, = 147456 + {\cdot}0009095869956096,$
$= 147456{\cdot}0009095869956096$, the square root of which is $384{\cdot}000001184$, will be the value of the third side.

Or :—

$384^2 + {\cdot}03^2, = 147456 + {\cdot}0009, = 147456{\cdot}0009,$
the square root of which is $384{\cdot}000001171$, will be the value of the third side.

In the one case, on the writer's hypothesis, we are carried more nearly to the radius, and in the other are less in excess of it, than on the Orthodox data; and it is consequently demonstrated that the part of the circumference of the circle contained by two radii is more nearly resolved into a straight line, on the writer's hypothesis, than on the Orthodox data; and must, if not absolutely true, be admitted to present the nearest approximation to the truth.

And now, Reader, if you think me wrong, let me ask you to say where, and how. I have bestowed much time and labour on this impeachment of what is supposed to be a truth beyond all doubt. What, after doubting, and examining, I now deny to be true, others also may at least doubt. If it be very truth, let it be proved to be so. The proof cannot be difficult; or, if it be so, should be so handled that it may be difficult no longer. To simplify the subject will be a public benefit. And as to doubt is a duty, where palpable proof is not supplied; so to supply such proof is also a duty—and one especially incumbent on that learned fraternity who claim to be regarded as our guides. In their hands, then, I leave it. I have done my duty. Let them do theirs.

APPENDIX B.

Paper read in the Mathematical Section of the British Association for the Advancement of Science, at their Twenty-ninth Meeting, held at Aberdeen, on Wednesday the 21st September, 1859.

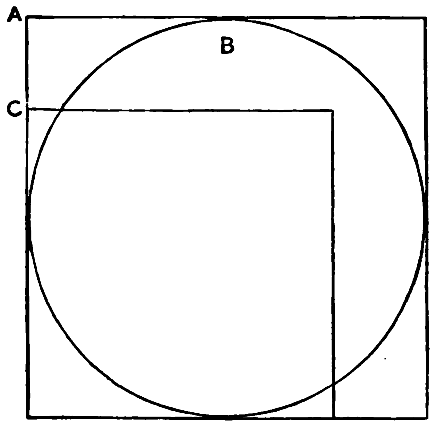
SIR WM. R. HAMILTON, LL.D., M.R.I.A., Astronomer-Royal of Ireland, in the chair.

ON THE RELATIONS OF A CIRCLE INSCRIBED IN A SQUARE.

IN drawing the attention of this section of the Association to the subject of "The Relations of a Circle inscribed in a Square," I shall confine myself simply to the development of a few facts. These facts appear to me of importance, and to be well worthy of consideration. I am satisfied there is contained within them the germ of truth, and that if further inquiry be instituted into the subject, it must lead to the discovery of other facts of great value to the advancement of science.

Without further preface I shall direct the attention of the section to the following diagram :—

Diagram No. 1.



In this diagram, let the side of square A be unity, and the side of square C equal in value to one-fourth of the circumference of circle B inscribed in square A.

Then, the diameter of circle B, inscribed in square A, must be unity. The area of circle B is said to be represented by the figures ·7854, and it necessarily follows that the ratio between the area of square A, and the area of circle B, will be, as 1 to ·7854.

The four sides of square A are together equal in value to 4. And, as 1 : ·7854 : : 4 : 3·1416, which is said to be the circumference of a circle of which the diameter is unity.

I am not aware that the following fact has ever been observed, viz. :—

That, as 1 : ·7854, so is the area of circle B to the area of square C, of which the side is equal in value to one-fourth of the circumference of circle B; or in other words, of which the four sides of the square are together equal in value to the circumference of circle B. Therefore, as 1 : ·7854 : : ·7854 : ·61685316, the area of square C. Then, $\sqrt{\cdot 61685316} = \cdot 7854$, must be the value of the side of

square C, and is equal in numerical value to the area of circle B.

It is admitted that the circumference of circle B is equal to four times its area, and the side of square C being equal in numerical value to the area of circle B, the circumference of circle B is also equal in value to four times the side of square C.

Again : as 1 : ·7854, so is the side of square A to the side of square C; or, as 1 : ·7854 : : 1 : ·7854, the side of square C.

This illustration may be made plainer and more striking by varying the figures.

Let me assume the side of square A to be 8, then the diameter of circle B inscribed in it must also be 8.

Now, the usual mode of ascertaining the circumference of circle B, is to multiply the diameter by the figures 3·1416, said to be the circumference of a circle of which the diameter is unity. Therefore, $8 \times 3{\cdot}1416 = 25{\cdot}1328$, will be the circumference of circle B; and half the circumference, multiplied by half the diameter, or, the square of the diameter multiplied by ·7854, will give the area of circle B. Therefore, $\frac{1}{2}(25{\cdot}1328) \times \frac{1}{2}(8) = 12{\cdot}5664 \times 4$, or $64 \times {\cdot}7854 = 50{\cdot}2656$, will be the area of circle B.

But these values may also be obtained by means of the ratios, as follows :

The four sides of square A are together equal in value to 32; and, as 1 : ·7854 : : 32 : 25·1328, the circumference of circle B.

The area of square A is 64, and, as 1 : 7854 : : 64 : 50·2656, the area of circle B.

But further : As 1 : ·7854, so is the area of circle B to the area of square C, of which the four sides are together equal in value to the circumference of circle B. Therefore,

as 1 : ·7854 :: 50·2656 :: 39·47860224, the area of square C. Then, $\sqrt{39{\cdot}47860224} = 6{\cdot}2832$, must be the value of the side of square C; and, as 1 : ·7854, so is the side of square A to the side of square C; or, as 1 : ·7854 : : 8 : 6·2832, the side of square C. Then, 4 times the side of square C, = 6·2832 × 4, = 25·1328, is the circumference of circle B. And it is demonstrated that the four sides of square C are together equal in value to the circumference of circle B, and that the ratio of 1 to ·7854 holds good in four distinct respects, as regards this diagram.

I may here remark, that if any other number than ·7854 be assumed to represent the area of a circle of which the diameter is unity, the only effect of it would be, to change the values of the circumference of circle B, and the side and area of square C, but would not, in any respect whatever, affect the principle involved in the ratios.

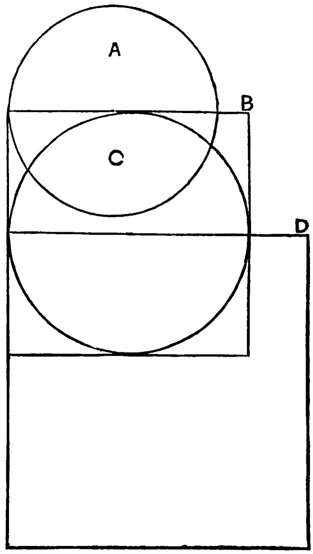
For example: If the area of circle B be assumed to be ·78125, then the ratio between the area of square A and the area of circle B would be as 1 to ·78125, and the effect of the alteration would be to give the value of the side of square C as ·78125, and its area ·6103515625, instead of ·7854 and ·61685316, as in my first illustration.

I must now direct the attention of the section to another diagram, in connection with what I have already adduced, and in doing so I shall assume the diameter of circle B, in diagram No. 1, to be unity, and its area to be ·78125.

In the diagram No. 2, let the diameter of circle A be the half of unity, or ·5.

Then, if circle A be circumscribed by a square, the value of the side of such square must be ·5, and the value of its area ·25. And, as 1 : ·78125 : : ·25 : ·1953125; or, ·25 × ·78125, is also equal to ·1953125, the area of circle A.

Diagram No. 2.



It is admitted that a circle described with any radius, is four times the value of a circle described with half that radius; and agreeably with this admitted proposition, the area of circle B, in diagram No. 1, is equal in value to four times the area of circle A, in diagram No. 2.

I must now direct the attention of the section to a fact, which I believe has never before been observed, viz.: that the ratio between the diameter of a circle and the side of a square, of which the area is equal in value to twice the area of the circle, is as 4 to 5.

Therefore, as 4 : 5, so is the diameter of circle A to the side of square B; or, as 4 : 5 : : ·5 : ·625, the side of square B. And, $\cdot625^2 = \cdot390625$, will be the area of square B, and is equal in value to twice the area of circle A.

Then, as 1 : ·78125, so is the area of square B, to the area of circle C inscribed in it; or, as 1 : ·78125 : : ·390625 :

·30517578125; or, ·390625 × ·78125, is also equal to ·30517578125, the area of circle C.

The diameter of circle C must be equal to the side of square B, = ·625. And, as 4 : 5, so is the diameter of circle C, to the side of square D, of which the area is equal in value to twice the area of circle C; or, as 4 : 5 : : ·625 : ·78125, the side of square D. And, $·78125^2 = ·6103515625$, will be the area of square D, and is equal in value to twice the area of circle C; and so we might proceed, *ad infinitum.*

At this point, I have to direct the special attention of the section to the following fact:—

That in the descending scale, from a circle of which the diameter is unity, and in the ascending scale, from a circle of which the diameter is the half of unity, the two meet in a square of precisely the same value, viz.: the square C, in diagram No. 1, and the square D, in diagram No. 2, a square of which the value of the side is ·78125, and its area ·6103515625.

I have now introduced to the notice of the section, in connection with this subject, facts which it is not in the power of anyone to contravert; and it is also certain that no other figures than those which I have assumed, viz., ·78125, as the area of a circle of which the diameter is unity, can produce the same results; and I have no hesitation in declaring that there are other facts beneath the surface of what I have stated, which only need to be sought for to be discovered, and when discovered, will be found to be of the utmost value and importance to the noble cause of science.

EDWARD HOWELL, PRINTER, CHURCH STREET, LIVERPOOL.

/6